Preservice teachers’ view on the concept of function

A study including the utilization of concept maps

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Abstract
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Foreword

Because it is literary impossible to master higher mathematics in any intellectually honest way without a firm and deep understanding of functions, mathematics educators are trying to identify and understand the learning obstacles students encounter in mastering this notion. (Eisenberg, 1992, p. 158)

I became engaged in preservice teachers’ learning in mathematics through my experiences working with teacher education. When I got the opportunity to pursue further studies in mathematics education I wanted to study preservice teachers’ view on the concept of function. In this thesis, preservice teachers’ conceptions of functions are mainly considered in relation to mathematical statements, related to different concepts and topics on a variety of levels in mathematics.

This thesis consists of an overview of the subject, where in particular the following three papers are put into a frame:


The overview part consists of: an introduction, a short description of some related research and theoretical framework, a summary of papers I-III and the concluding discussion.
1 Introduction

1.1 The function concept

The concept of function is fundamental in mathematics. According to the contemporary definition, a function is a correspondence between two non-empty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). In the historical development of the concept Peter Gustav Lejeune Dirichlet was the first to take serious notion of this definition during the first half of the nineteenth century, at a time when mathematicians “in practice … thought of functions as analytical expressions or curves” (Kleiner, 1989, p. 291). But a common consensus of the definition (Malik, 1980; Monna, 1972) was not established in the mathematics community until the first half of the twentieth century when Nicolas Bourbaki more firmly established the definition, and it is often called the Dirichlet-Bourbaki\(^1\) definition (Kleiner, 1989; Rüthing, 1984; Youschkevich, 1976).

The definition of function simply utilizes the idea of univalence, that for each element in the domain there is exactly one element in the codomain, with no other required properties of the correspondence. Univalence in combination with domain and codomain as two arbitrary chosen nonempty sets makes the concept of function a highly general and abstract notion that proves to be demanding for students to assimilate (Accoc & Tall, 2003; Eisenberg, 1991; Even, 1993; Sierpinska, 1992; Tall, 1992; Vinner & Dreyfus, 1989). Moreover, the concept of function has several synonyms such as mapping, operator, transform etc. that are used in different contexts and in various forms of representation. It has furthermore an extensive set of sub-concepts and a large network of relations to other concepts (Blomhøj, 1997; Eisenberg, 1991, 1992; Selden & Selden, 1992; Tall, 1992, 1996) that makes assimilation of the function concept and an understanding of its significance a long-term process for students in mathematics.

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\(^1\) The Bourbaki group defined in 1939 a function as a correspondence between two sets in a similar manner as Dirichlet had done 1837 (Monna, 1972; Stigler, 1984; Youschkevitch, 1976). Bourbaki also formulated an equivalent definition of function as a set of ordered pairs (Kleiner, 1989), where a function from a set E to a set F is defined as a special subset of the Cartesian product \(E \times F\). Bourbaki’s view of function differs from Dirichlet’s view in that the domain and codomain no longer are restricted to sets of numbers. In Dirichlet’s definition of function the domain is a finite interval of real numbers and the codomain consists of the real numbers.

Dirichlet was the first one to take serious notion of functions that are consistent with the modern definition. He gave in 1829 at the end of a paper on Fourier series the Dirichlet function (a function with domain \([0, 1]\) and codomain \([0, 1]\) that assigns rational numbers to 0 and irrational numbers to 1, Malik, 1980; Youschkevitch, 1976), as an example consistent with the modern definition of function neither possible to represent as a curve nor an analytical expression. Dirichlet’s paper from 1837 (Dirichlet, 1837) with his definition of function is an elaborate version of the paper from 1829 (Dirichlet, 1829), according to Hawkins (1975).
1.2 A unifying concept, network of relations and school mathematics

The concept of function is part of all areas in mathematics and frequently considered as the unifying concept of mathematics. The significance of functions is manifested by their large network of relations to other concepts, and is for many concepts part of the definition. To develop an understanding of the function concept includes a comprehension of this network of relations. To gain knowledge of the relations and to be able to use functions in different contexts is a learning process that requires a longer period of time. It is therefore appropriate to introduce the concept of function in school mathematics, to be able to gradually expand the students’ knowledge of functions, their applications, representations and relations to other concepts, and successively making the students able to handle functions in a more flexible way.

Historically there have been initiatives to emphasize the concept of function in pre-tertiary education. According to Cooney and Wilson (1993) many mathematics educators during the early 20th century believed there was a need for greater emphasis on functional thinking in school mathematics (referring to, Breslich, 1928; Hamley, 1934; Hedrick, 1922; Schorling, 1936). The well-known mathematician Felix Klein became engaged in mathematics education and played a central role in a curriculum reform, the “Meran Programme”2, declared in 1905 (Cooney & Wilson, 1993; Fujita, Jones & Yamamoto, 2004; Sierpinska & Lehrman, 1996). The concept of function as a dependency relation was a central part of the reform and was considered as a unifying concept in mathematics. A similar development was seen in the curriculum in other countries, affected by the Meran Programme. But the effect of the reform was not noticeable (Cooney & Wilson, 1993; Sierpinska & Lehrman, 1996), and Cooney and Wilson (1993) question if the emphasis on functions in reality reached as far as compulsory school, and suggest that one reason could be teachers’ conceptions of functions. The concept of function was also advocated in the new mathematics movement of the 1960s, but this time usually in a more formalistic approach as a set of ordered pairs, that later have been considered a less appropriate way to introduce the function concept, in school (Cooney & Wilson, 1993; Eisenberg, 1991; Tall, 1992, 1996).

It is important for mathematics teachers in becoming successful dealing with the concept of function in their practice to have a well-developed conceptual knowledge of functions, including the concept’s significance in mathematics and relations to other concepts (Cooney & Wilson, 1993; Eisenberg, 1992; Even, 1993; Thomas, 2003; Vollrath, 1994). The concept of function is currently a regular part of the school-mathematics curriculum. In Sweden functions are introduced at compulsory school where some basic

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classes of functions are considered, and typically a dependency relation between two variables stressed and the term “function” less often used. At upper-secondary school the concept of function and its definition is more explicitly stated, where the standard functions are part of courses in introductory calculus required for further studies in mathematics at tertiary level.

1.3 Preservice teachers and aims of research

The aims of research in my study concern preservice teachers’ view on the concept of function at the end of their required courses in mathematics, on a teacher preparation program in mathematics and science for the school grades 4 to 9. The preservice teachers’ conceptions of functions and their different properties are mainly examined in relation to mathematical statements where the study successively has been expanded to include \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \). The three mathematical statements can be related to mathematical concepts and different topics on a variety of levels, associated with the preservice teachers’ future teaching as well as concepts and topics at tertiary level. In Paper III of this thesis, I present some further contributions considering this topic. Among other concepts \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) can be recognized as examples of different classes of real functions of a real variable – such as for example a linear, quadratic or rational function, respectively – in using a form of representation not uncommon regarding the concept of function (Eisenberg, 1991, 1992; Tall, 1996; Yerushalmy, 1997). The study aims to examine preservice teachers’ understanding of functions in this context. A more detailed account of the research questions is given in the beginning of section four.

2 Related research

2.1 Studies related to the concept of function

The significance of the function concept in mathematics is reflected by a substantial research literature regarding the concept of function in mathematics education. Where conferences dedicated to the function concept generate books such as The Concept of Function: Aspects of Epistemology and Pedagogy (Harel & Dubinsky, 1992) or Integrating Research on the Graphical Representation of Function (Romberg, Fennema & Carpenter, 1993).

A number of studies have been conducted about students’ conceptual knowledge of function at tertiary level, confirming a frequent inconsistency in students’ conceptions of function and the definition of function (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Cuoco, 1994; Eisenberg & Dreyfus, 1994; Even, 1990, 1993, 1998; Romberg, Carpenter & Fennema, 1993; Slavit, 1997; Tall & Bakar, 1992; Thomas, 2003; Thompson, 1994;
Vinner & Dreyfus, 1989; Williams, 1998, to name a few). One such well known study was conducted by Vinner and Dreyfus (1989) showing that tertiary students during a course in calculus, even in the case when the students were able to give a correct account of the definition of function, did not apply the definition of function successfully. Vinner (1983, 1992) describes a model consistent with these results regarding the definition of function (in form of concept image) further described in section three. Similar results have been reported in a range of studies (e.g., Tall & Bakar, 1992; Breidenbach et al., 1992; Even, 1993; Thomas; 2003) where Breidenbach et al. (1992) points out that “college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept” (p. 247), confirming that it is a complex concept for students to grasp and that the conceptual development requires a longer period of time.

A majority of researchers in the community of mathematics education (Confrey & Doerr, 1996; Eisenberg, 1991, 1992; Dubinsky & Harel, 1992; Freudenthal, 1983; Romberg, Carpenter et al., 1993; Selden & Selden, 1992; Sierpinska, 1992; Sfard, 1992; Tall, 1992, 1996; Yerushalmy & Chazan, 2002, to name a few) seems to agree that the concept of function should be introduced in a dynamic form, such as a type of relation, correspondence or covariation – not favoring a static ordered pair version of the definition, related to Bourbaki. Researchers are then stressing a number of different approaches representing different theoretical frameworks to develop students’ conceptual knowledge of function such as modeling, programming, multiple representations etc. to successively develop students conceptual understanding and first at tertiary level, use the definition of function in its full generality when it is required in the study of more advanced topics.

In students’ conceptual development of function process-object models are frequently suggested (Dubinsky & Harel, 1992; Eisenberg, 1991; Selden & Selden, 1992; Sfard, 1992; Tall, 1992, 1996; Thompson, 1994), and the concept of function is often used to illustrate conceptual development in different theoretical models. Including the well-known models by Sfard (1989, 1991, 1992) – further described in section three – and Dubinsky with colleagues (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996; Breidenbach et al., 1992, Dubinsky, 1991; Dubinsky & Harel, 1992) known as theory of reification and APOS theory, respectively, and the theory of pro- cepts (Gray & Tall, 1994) underlining the role of symbols (the three theories are further discussed in, e.g. Mamona-Downs & Downs, 2002; Tall, Thomas, Davis, Gray & Simpson, 2000).

While models of conceptual development using “object” as a central metaphor are frequent they have also been criticized (e.g., Confrey & Costa, 1996; Dörfler, 1996). With requests to clarify the meaning of a mathematical object (Godino & Batanero, 1998) and alternative frameworks suggested, for
example prototypes (Akkoc & Tall, 2002; Schwarz & Hershkowitz, 1999; Tall & Bakar, 1992), multiple representations (Borba & Confrey, 1996; Kaput, 1992; Keller & Hirsch, 1998) or combinations of frameworks into broader perceptions referring to “horizontal growth” in different forms of representation and “vertical growth” in the development from process to object (Schwingendorf, Hawks & Beineke, 1992, or analogous models, such as DeMarois & Tall, 1996, using the terms “facet” and “layer”).

2.2 Teachers’ knowledge, preservice teachers and functions

Even if there is a considerable and growing body of research on the nature and learning of the function concept most of this research has been focusing on students’ conceptions of function. Only a minor part of the research has addressed teachers’ or preservice teachers’ cognitions and appropriate knowledge of functions (Chinnappan & Thomas, 1999, 2001; Cooney & Wilson, 1993; Even, 1990, 1993, 1998; Even & Markovits, 1993; Grevholm, 2000a, 2000b; Leikin, Chazan & Yerushalmy, 2001; Haimes, 1996; Norman, 1992; Thomas, 2003, to name a few). The idea that a teacher’s content knowledge base will influence the quality of the understanding that students develop in an area of mathematics has received support from research findings (Fenemma & Franke, 1992; Even & Markovits, 1993; Even & Tirosh, 1995, 2002). This is not particularly surprising since one might expect both lesson goals and structures to be contingent on teacher understanding of the subject matter.

Shulman (1986) introduced the concept of pedagogical content knowledge for these types of activities, suggesting the existence of links between content knowledge and explanations and representations generated during teaching. The distinction between being able to apply a relatively well determined set of instructions to a mathematical problem and being able to explain why doing so, is of crucial importance in this context. Skemp (1976, 1978) makes the distinction between instrumental (knowing how) and relational (knowing how and why) understanding, and why teaching and learning in mathematics risks to promote instrumental instead of relational understanding.

The notions of knowledge and understanding are multidimensional and described in various forms in the mathematics education literature (Even & Tirosh, 2002; Sierpiska, 1994). Skemp’s instrumental and relational understanding is for example largely mirrored by procedural and conceptual knowledge by Hiebert with colleagues (Hiebert & Lefevre, 1986; Hiebert & Carpenter, 1992). Where procedural knowledge is a form of sequential knowledge constructed in a succession of steps, and conceptual knowledge may be considered as a well-connected web of knowledge for flexibly accessing and
selecting information. Both kinds of knowledge are required for mathematical expertise, according to Hiebert.

Studies regarding preservice teachers’ conceptual knowledge of function are limited, and even more so on topics as regards preservice teachers’ learning and future teaching (e.g., Chinnappan & Thomas, 2001; Cooney & Wiegel, 2003; Doerr & Browsers, 1999; Even, 1990, 1993, 1998; Grevholm, 2000a, 2000b; Sánchez & Llinares, 2003; Wilson, 1994). Studies like Leikin, Chazan and Yerushalmy (2001) and Thomas (2003) show that inservice teachers’ conception of function is not consistent with the definition of the concept. Even (1993) stresses the importance that (secondary) mathematics teachers have a concept image of function that is consistent with the contemporary definition of function. Emphasizing an understanding on functions “arbitrary nature”3 (p. 96) and an understanding of the requirement of univalence, and an ability to use different representations (Even, 1998).

Even (1993) also calls to attention that courses in mathematics for preservice teachers should be constructed so that a better, more comprehensive and articulated understanding and knowledge of functions (and mathematics) is developed. Vollrath (1994) suggests concepts as starting points for didactical thinking in mathematics. He sees discussions about “central concepts” (p. 63) as an essential part in courses for preservice teachers, and considers knowledge about the importance of concepts their use and relations to other concepts as vital for teachers’ planning and teaching in mathematics. Eisenberg (1992) describes what he calls “having a sense for functions” (p. 154) as a major goal in the curriculum, describing this notion as having insights about functions that incorporates the integration of many skills. Skills often taught in isolation where compartmentalization of knowledge risks occurring when a body of knowledge splits into a larger number of isolated bits (using parts of the theory of Chevallard’s didactical transposition, Chevallard, 1985, which concerns the change knowledge undergoes as it is turned from scientific, academic knowledge to instructional knowledge as taught in school). More concerns regarding knowledge compartmentalization is considered in students not being able to assimilate different forms of representations of functions (Leinhardt, Zaslavsky & Stein, 1990; Mamon-Downs & Downs, 2002), with impact on understanding, facility in manipulation, mental imagery etc.

2.3 Concept maps and the concept of function

Few studies have been conducted using concept maps in mathematics education – see Paper II for further details – with only a minor part accessing stu-

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3 Even (1993) points out two essential features that have evolved in the definition of function: arbitrariness and univalence (referring to Freudenthal, 1983). The arbitrary nature of function refers to both the relationship between the two sets on which the function is defined and the sets themselves.
students’ conceptual knowledge of function (such as, Doerr & Brown, 1999; Grevholm, 2000a, 2000b; Leikin et al., 2001; McGown & Tall, 1999; Williams, 1998). Williams (1998) gives an indication of how differentiated conceptual knowledge of functions are expressed in concept maps, comparing concept maps of professors having PhDs in mathematics with first-year university students taking a course in calculus. Williams found for example that many students’ concept maps contained trivial parts and parts of an algorithmic nature. In contrast, the professors’ concept maps reflected many properties, categorical groupings, function classes and common types of functions. None of the professors’ maps demonstrated the students’ inclination to think of a function as an equation. Instead, they defined it as a “correspondence, a mapping, a pairing, or a rule” (p. 420). Concept maps are also used in studies concerning students’ conceptual development, like McGown and Tall (1999) to document the process by which college students construct, organize and reconstruct their knowledge about functions, during a course in algebra. They are concluding that high performing students build rich conceptual frameworks on anchoring concepts that develops in sophistication and power, whereas lower achievers reveal few stable concepts with conceptual frameworks that have few stable elements.

Concept maps have also been used to study preservice mathematics teachers’ conceptual development, such as Grevholm (2000a, 2000b) in a longitudinal study of preservice teachers’ conceptual development including the concepts of equation and function. Early results are presented which indicate that preservice teachers’ cognitive structure slowly develop to become a clearer and richer structure. Doerr and Bowers (1999) use concept maps as a tool to study preservice teachers’ conceptions of the function concept, related to students learning. The concept maps show that the concept of function is largely disconnected from pedagogical strategies or learning paths that students might encounter. However, later after a course designed to challenge the preservice teachers’ knowledge about learning and the concept of function, such knowledge is integrated with their understanding of the function concept, thus calling attention to such activities.

3 Theoretical framework

There is a range of theoretical frameworks in the field of mathematics education (Lerman & Tsatsaroni, 2004; Niss, 1999; Sierpinska, 2003). When it comes to the concept of function I believe the observation made by Eisenberg is still valid:

A major obstacle in discussing the learning problems associated with functions in particular … is that there is no generally accepted theoretical framework as a basis for discussion. (Eisenberg, 1991, p. 142)
and a number of different frameworks are being used, as noticed in section two.

Ausubel (1968, 2000; Ausubel, Novak & Hanesian, 1978) describes learning in terms of “meaningful learning” as opposite to “rote learning” with consequences to linkages in a model of an individual’s cognitive structure where previous knowledge is essential in the learning of new knowledge. A similar approach can be found in mathematics educational literature by Hiebert and Carpenter (1992) using network metaphors referring to Ausubel’s theories as an “bottom-up approach”, in the way knowledge structures develop building upon prior knowledge, and a view of “learning with understanding” similar to Ausubel’s “meaningful learning” in the formation of internal dynamic knowledge structures where structure and linkage is vital for understanding. In Papers I and II of this thesis further contributions in this connections are presented. This kinds of network based models are common in different fields related to learning such as cognitive science, artificial intelligence and neuro science (Anderson, 2000; Baddeley, 1997; Gärdenfors, 2000). They are also used in more general frameworks in mathematics education as a support for learning such as “webbing” by Noss and Hoyles (1996).

In the chosen framework knowledge is represented internally and understanding is described in terms of the way an individual’s mental representation is structured. Internal representations can be linked, metaphorically, forming dynamic networks\(^4\) of knowledge with different structures, especially in forms of webs and vertical hierarchies (Hiebert & Carpenter, 1992). Understanding grows as the networks become larger and more organized where existing networks influence relationships that is constructed thereby helping to shape the new networks that are formed. The construction of new relationships may force a reconfiguration of affected networks. Ultimately, understanding increases as the reorganizations yield more richly connected, cohesive networks. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak.

Vinner (1983, 1992) describes a model for the correspondence between the definition of function and an individual’s understanding of the concept, applicable to formal concepts in general (Vinner, 1991). The key idea is the distinction between concept definition and concept image. The concept definition concerns a form of words used to describe a concept (it may be a personal concept definition different from a formal definition accepted by the mathematics community). The concept image refers to the total cognitive structure in the mind of an individual that is associated with the concept, in-

\(^4\) The network model is a common model of human memory in cognitive psychology (Anderson, 2000; Baddeley, 1997) were some models, like Ausubel’s assimilation theory (Ausubel, 2000), makes further assumptions about semantic memory being hierarchal etc.
cluding “all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). A concept image is built up during a longer period of time through an individual’s experiences of all kinds. The portion of a concept image that is activated at a particular time is called the evoked concept image. In thinking, almost always the concept image will be evoked, where the concept definition will remain inactive or even be forgotten. When students meet an old concept in a new context, it is the concept image with all the implicit assumptions abstracted from earlier contexts that respond to the task.

In students’ conceptual development of function Sfard (1989, 1991, 1992) suggests a process-object model, where the formation of an “operational” conception of function as a process precedes a later more mature phase in the formation of a “structural” conception regarding functions as objects. Both conceptions are essential and should coexist forming a dual view of the concept, according to Sfard. In the transition from operational to structural conception a three-step pattern emerges: interiorization, condensation, and reification. Reification is the final step giving an individual the ability to conceptualize a concept as an object. Without reification an individual’s conceptual understanding will remain purely operational. In the process of developing a structural conception of function Slavit (1997) suggests an emphasis on functions’ properties to enhance the phase of reification, where the notion of invariance is significant (Bagni, 2003).

4 Summary of the papers

The three papers Hansson (2004a, 2004b) and Hansson and Grevholm (2003) concerns preservice teachers on a four and a half year long teacher preparation program in mathematics and science for the grades 4 to 9. The papers are part of a larger study located to the sixth term during the program’s concluding courses in mathematics, with the exception of Hansson and Grevholm (2003) that also includes preservice teachers in their third term. The sixth term contains a course in calculus where the concept of function is central, and the preservice teachers’ view on functions is primarily considered after the calculus course.

4.1 Aims of the study

The aims of research are related to preservice teachers’ conceptions of function and mainly examined in relation to mathematical statements, where the study successively has been expanded to include \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \). The three mathematical statements can be linked to different concepts and topics on a variety of levels. They can in particular be recognized as examples of
real functions of a real variable. The aims of the study are to answer the following questions:

Paper I: What conceptions do preservice teachers have and what is their concept of function in connection to \( y=x+5 \)? What progression can be seen between two groups in their third and subsequently their sixth term, in a teacher preparation program?

Paper II: The aim is to investigate the use of concept maps to reveal preservice teachers’ knowledge and understanding of the concept of function in relation to the mathematical statements \( y=x+5 \) and \( y=\pi x^2 \). More specifically, the questions of research are: In what way do preservice teachers construct concept maps starting with the mathematical statements \( y=x+5 \) and \( y=\pi x^2 \) respectively? How is the concept of function expressed in the maps? What knowledge is displayed and what qualities are desirable in such a map? What experiences of drawing the maps do the preservice teachers express?

Paper III: What conceptions of function do preservice teachers have in relation to \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \) concerning preservice teachers on different levels of performance in their studies in mathematics? What properties do they notice and how are relations between the concept of function and other concepts described, in relation to the mathematical statements?

4.2 Methods and design of the study
The teacher preparation program includes courses in mathematics for 30 weeks full time study, where approximately one third is related to mathematics education. The courses are distributed to the first, third and sixth terms. The groups participating in the study involve the total number of preservice teachers in the teacher preparation program who were specializing in mathematics and science, for each term the study takes place.

4.2.1 Paper I: Preservice teachers’ conceptions about \( y=x+5 \): Do they see a function?
The study included two groups of preservice teachers who were in the third and sixth terms of the teaching training program, respectively. It was initiated by Grevholm (1998, 2002) with a group of 38 preservice teachers who took an algebra course in their third term. The study was repeated three and a half years later by Hansson (2003), with a group of 19 preservice students who enrolled in a calculus course in their sixth term.
A survey including an open question on \( y = x + 5 \) was used in the study. The answers to the questions were categorized according to the categorization made by Grevholm (1998, 2002). The questionnaire was distributed before and after each course, during a mathematics lecture; the answers were then compiled for each group of preservice mathematics teachers. In order to guarantee that the groups’ answers were divided into categories in a standardized manner, the authors separated the answers from both groups into the six categories independently of each other. The results were compared and discussed and the procedure repeated until the division was such that the distribution of the answers from the algebra group corresponded with that which was presented in Grevholm (1998, 2002). Examples of how the answers where divided into categories are given in Paper I.

The preservice teachers’ survey answers were often brief. Eight students were thus interviewed after the algebra course in order to obtain a better understanding of their views on \( y = x + 5 \). The interviews were recorded on tape. The students were selected based on the answers they gave in the survey. Further seven students were interviewed after the calculus course; four of these interviews were recorded on tape, while notes were taken during the other three interviews. The interviews were conducted on the basis of the preservice teachers’ answers to the survey and they were given a chance to further expand on their answers. The interviews and the transcription of the answers were conducted by Grevholm for the students in the third term and by Hansson for the students in the sixth term (excerpts of the interviews are disclosed in Grevholm, 1998, 2002, and Hansson, 2003).

In order to continue studying how preservice teachers view the statement \( y = x + 5 \), Hansson used concept maps in an unorthodox manner by allowing the sixth-term students to draw concept maps which were derived from \( y = x + 5 \) after the calculus course. A more detailed description of the procedure and the manner in which the maps were analyzed is presented below, in the description of the method for Paper II.

### 4.2.2 Paper II: An unorthodox utilization of concept maps for mathematical statements: Preservice teachers’ response to a diagnostic tool

Two groups of preservice teachers participated in the study. The study was conducted over a period of two years, when each group was in the sixth term of the teaching program and had completed the course in calculus.

There are several studies in mathematics education that involve various types of maps, all of which are referred to as “concept maps” (see Paper II). It is in agreement with the aims of the study to investigate how the preservice students construct different types of concept maps in relation to the mathe-
mathe-atical statements, as well as how they describe the experience of drawing these maps. I have chosen to investigate how the students construct two common types of concept maps: concept maps with a freely chosen structure and those with a hierarchical structure.

The first group consists of 19 students (the same group as in Paper I). The preservice teachers were introduced to concept maps during a lecture related to mathematics education; the process is described in more detail in Paper II. They were then each instructed to draw an individual concept map for y=x+5 in the manner of their choise. A week later, the preservice teachers drew a new map for y=x+5; this time, however, they were instructed to construct the map in a hierarchical format. On a third occasion, they were given an opportunity to comment on their maps by answering a set of questions which are further discussed in Paper II.

All of the concept maps were analyzed. Each map was analyzed as an integrated unity in which the contents and structure of the map were noted. Furthermore, the manner in which the different sections of the map were related to each other was also studied. In particular, the manner in which the function concept was expressed on the maps was noted; its relationship to other concepts and properties which were assigned to functions were considered. The contents of the maps, the preservice teachers’ answers to the questionnaire and their comments on the maps were also compiled in tables. A primarily quantitative analysis and a method of categorization based on the contents of the nodes and the number of links connecting them to other nodes was also tested. The quantitative analysis was abandoned in favor of a more qualitative analysis of the maps, since it seemed that valuable information about how different parts of the map was connected and how relations between different concepts were described was lost.

The study was repeated with a group of 25 preservice teachers the following year. Based on an analysis of maps, comments from the previous year’s participants and the experienced gained, the hierarchical maps appeared to have greater potential for providing information on the preservice teachers’ perception of the function concept in relation to y=x+5. At the same time, this type of map was considered to be more demanding for the students to draw and was therefore often less detailed than the map that had been freely designed. The result of this was that the students who participated in the study the following year drew the maps on one occasion and were directed to begin with the freely formatted map before constructing the hierarchical map. Furthermore, the study was expanded to include y=πx^2. An analysis of the maps was conducted in the same manner as in the previous year.
4.2.3 Paper III: Preservice teachers’ view on three mathematical statements: A case study regarding the concept of function

A group of 25 preservice teachers (the same group as described in Paper II) participated in the study when they were in the sixth term of the teaching program. A questionnaire\(^5\) was distributed before and after the calculus course, during a lecture in mathematics. The questionnaire contained questions which asked the preservice teachers to describe a function, in addition to answering open questions in relation to the statements \(y=x+5\), \(y=\pi x^2\) and \(xy=2\). In order to study their views on the three statements and ask questions about their views on them, twenty students from the group were interviewed. The interviews were based on the answers which the preservice teachers had given on their two questionnaires, in which they were given the opportunity to comment on and expand their answers, draw pictures, etc. (questions which will be used in a continuation of this study were also asked during the interview). The preservice students also drew concept maps\(^6\) for \(y=x+5\) and \(y=\pi x^2\) after the calculus course, as described in Paper II.

Since the case study in Paper III is based on the interviews, the method and the practical arrangements will be described in more detail. The students who were willing to be interviewed signed up for participation by writing their names on a timetable. The interviews were conducted separately in a preparation room for mathematics during a three-week period. The length of the interviews varied, but often lasted for an hour or more and touched on all the questions on the questionnaire. If the student did not mention the function concept in relation to the three statements, then I could ask them to comment on the statement in view of the previous questions in the survey, in which the function concept is discussed. The interview was based on the preservice teachers’ answers to the survey, which they were asked to comment on. However, to obtain a better understanding of the preservice teachers’ thought processes I also asked questions that were tied to their answers. An interview normally took place during a predetermined time interval, which meant that I sometimes had to move on to the next question.

Brief notes were usually made after an interview to summarize the results. Each interview was recorded on tape and personally transcribed according to recommendations made by Kvale (1996) (this is described in more detail in

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\(^5\) The actual survey differs from that which was used in Paper I, with the exception of the introductory questions, which is mainly the same in the two surveys. This has been done to create similar conditions for the groups of students which attended calculus courses when they answer the open question on \(y=x+5\), in case a comparison should be necessary. Explicitly formulated questions on the function concept are also included in the survey of which some are intended to be included in a continuation of the study. The survey can be found in the Appendix.

\(^6\) To draw concept maps is a time consuming process, and there was not an opportunity to draw concept maps for all three mathematical statements.
of collected is appropriate material and activities for the students. Already offered Törner, interactive, Learning Experiences, ideas rather knowledge knowledge (Ernest, 1996; Phillips, 1995; Steffe & Gale, 1995). Knowledge is an individual construction built gradually. Understanding grows as an individual’s knowledge structures become larger and more organized, where existing knowledge influence relationships that is constructed; understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. To promote understanding includes critical dimensions in mental activities such as constructing relationships, extending and applying mathematical knowledge, reflecting about experiences, articulating what one knows, etc. (Carpenter & Lehrer, 1999). Learning becomes meaningful when the learner chooses to relate new knowledge to prior knowledge, nonarbitrary and substantive (Ausubel, 2000). Learning is not just a passive absorption of information. Rather it is more interactive, involving the selection, processing and assimilation of information of the learner, also affected by values and believes (Leder, Pehkonen & Törner, 2002). In environments that promotes learning students should be offered a broad range of teaching strategies, taking into account what students already know, presenting concepts and general ideas, and attending to appropriate material and activities for the students.

According to the theoretical framework, where an individual’s knowledge is represented internally, the methods used in the study emphasis on data collected individually to examine the preservice teachers’ different conceptions of function. The research design includes both quantitative and qualitative

4.2.4 Methodological discussion

The theoretical framework is considered to belong to the constructivist genre (Ernest, 1996; Phillips, 1995; Steffe & Gale, 1995). Knowledge is an individual construction built gradually. Understanding grows as an individual’s knowledge structures become larger and more organized, where existing knowledge influence relationships that is constructed; understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. To promote understanding includes critical dimensions in mental activities such as constructing relationships, extending and applying mathematical knowledge, reflecting about experiences, articulating what one knows, etc. (Carpenter & Lehrer, 1999). Learning becomes meaningful when the learner chooses to relate new knowledge to prior knowledge, nonarbitrary and substantive (Ausubel, 2000). Learning is not just a passive absorption of information. Rather it is more interactive, involving the selection, processing and assimilation of information of the learner, also affected by values and believes (Leder, Pehkonen & Törner, 2002). In environments that promotes learning students should be offered a broad range of teaching strategies, taking into account what students already know, presenting concepts and general ideas, and attending to appropriate material and activities for the students.

According to the theoretical framework, where an individual’s knowledge is represented internally, the methods used in the study emphasis on data collected individually to examine the preservice teachers’ different conceptions of function. The research design includes both quantitative and qualitative
elements. Furthermore, data is gathered to make triangulation possible, to enhance the validity of the findings and offer a more complete picture of the preservice teachers’ conceptions about the function concept. I was the instructor during the concept mapping and conducted the interviews. This ensured that the treatment was applied as designed. I was known as a teacher by the preservice teachers, which facilitated natural interactions in the interview. I have frequently used codes to represent individuals rather than names in the compilation of data; one reason is to prevent that my previous knowledge about the preservice teachers influenced the data analysis.

The concept maps are an essential part of the collected data and are used to study conceptual relationships (Duit, Treagust & Mansfield, 1996; Williams, 1998; Novak, 1998; Novak & Gowin, 1984), and to investigate preservice teachers’ conceptual understanding in a similar manner as the studies presented in section 2.3. But in contrast with those studies the concept maps in the current study start with a mathematical statement (as described in Paper II) to further examine the preservice teachers’ different conceptions related to the mathematical statements and the concept of function.

The interviews, as described in 4.2.3, made it possible to further study the preservice teachers’ view on the concept of function. I informed the interviewee that I was interested in how she or he thought and reasoned. The interviewee studied the two questionnaires from before and after the calculus course (the questionnaires were handed in at each occasion), and was asked to comment the answers. A preservice teacher was thus given the opportunity to reflect upon and comment the answers for each question on the questionnaire, and hence given an opportunity to contemplate on how the answers had changed, eventually, during the calculus course where the concept of function was a central concept. I asked further questions, probe and follow up questions (Kvale, 1996, 1997), in relation to the preservice teacher’s answers to understand how the preservice teacher was reasoning.

4.3 Main results

4.3.1 Paper I: Preservice teachers’ conceptions about y=x+5: Do they see a function?

The written answers show a similar development in both groups in that the preservice teachers use to a higher degree a numerical interpretation of y=x+5 before the course, which decreases after the course with a growth in linear and functional interpretation with the existence of two variables as a large and rather stable category. For a majority of preservice teachers, in both groups, the concept of function is not evoked in connection to y=x+5. The preservice teachers seem to have a tendency to use mathematical knowledge on a less advanced level than they have worked with in their courses in mathematics.
More developed views upon the function concept, as an object with many properties, are hardly visible. This became apparent in the written answers but also clearly in the concept maps. The concept maps contain more information than the written answers, and they are more developed in the area of a straight line. It seems that the preservice teachers’ concept of function is not represented by a rich cognitive structure in the context of \( y=x+5 \).

Examining data for individual students confirm that concepts develop slowly. More than every second student gives answers in the same categories before and after the course. Other students just add an extra category. The group of preservice teachers who had progressed further in the teacher preparation program had a slightly more elaborated language (but not yet elaborate enough to become successful as an inservice teachers) and flexible way of looking at \( y=x+5 \), where for example the concept of equation was more common.

4.3.2 Paper II: An unorthodox utilization of concept maps for mathematical statements: Preservice teachers’ response to a diagnostic tool

The preservice teachers construct their concept maps in a wide range of different styles when they draw concept maps as they find suitable. There is a tendency to let the mathematical statement become a hub surrounded by other parts of the map, where the different parts have varying degrees of web structure with few tendencies to become hierarchical. Even if the preservice teachers concept maps has numerous links they do not always give the impression to illustrate meaningful relations between different concepts. The hierarchical maps became, for natural reasons, more homogenous in structure. When the preservice teachers in the first group constructed hierarchical maps apart from the non-hierarchical maps, the hierarchical maps contained fewer links and the mathematical statement was to a higher degree linked to concepts it was deemed to represent. When the second group constructed their concept maps at the same occasion there was a higher correlation in both structure and content between the concept maps. The preservice teachers usually mix general concepts with more specific concepts when they draw hierarchical maps.

The concept of function is not a well-integrated concept and rarely a well-developed concept in the preservice teachers’ maps, indicating that the preservice teachers’ concept of function is not represented by a rich cognitive structure. When the preservice teachers add a concept to the concept map it is less common that they notice its relation to the concept of function. An illustration of this is for example a map showing that \( y=\pi x^2 \) is understood to represent a parabola with a minimum point, where the map shows that \( y=\pi x^2 \) also
is recognized as a function. However, there are no connections between the concept of parabola and the concept of function showing that the parabola is a function graph or that the minimum-point corresponds to a minimum-value of the function. The preservice teachers’ view of the function concept, as it appears in the concept maps, clearly diverges from the idea that functions play a central and unifying role in mathematics. Moreover, the concept maps usually do not express a more developed structural view of the concept of function as an object with a set of properties. The concept of function is often expressed as an operational conception in form of a dependency relation between variables where some preservice teachers state the univalence requirement, that an x value gives one y value, but no one mentions domain or codomain as parts of the function concept. The fact that the preservice teachers had just taken a course in calculus where the concept of function is a central concept, with a range of different properties and classes of functions, was rarely noticeable in the concept maps.

The concept maps indicate that different concepts – including the concept of function – which the preservice teachers relate to the mathematical statement the concept map begin with, tend to be separated and thereby preventing preservice teachers from building a conceptual framework rich of meaningful connections. It is not unusual that the preservice teachers’ concept maps contain trivial parts not always related to mathematics, at the expense of relevant mathematical concepts and relations between concepts, with indication of root learning and a less developed understanding of conceptual relations in mathematics. There are also parts expressing procedural knowledge and knowledge of an “algorithmic nature”; if for example the concept of derivative is part of the map it is in the context of finding stationary points, and not as a property of the function. Moreover, the concept maps show that a large part of the preservice teachers use mathematical terminology infrequent or incorrect, which could be an obstacle for meaningful learning. Furthermore, even if the mathematical statements give the preservice teachers an opportunity to relate to future teaching such themes are surprisingly rare in the concept maps. Using evaluation principles influenced by Ausubel’s assimilation theory the utilization of concept maps seems to reveal important aspects of individuals’ knowledge and understanding in relation to the given mathematical statements.

In the preservice teachers’ response to the activity of drawing concept maps there are signs of metacognitive activity and mediation, but also indication of an ability to evoke concept images with conflicting pieces of information; where hierarchical maps seem more mentally demanding to draw. Hierarchical maps also appear to have higher potential as diagnostic tools in the given context.
4.3.3 Paper III: Preservice teachers’ view on three mathematical statements: A case study regarding the concept of function

None of the preservice teachers in the case study describes the concept of function in a way that is consistent with the definition of function. As the preservice teachers’ performance in their studies in mathematics decrease, an operational understanding (Sfard, 1989, 1992) of the concept of function becomes more prominent. Knowledge structures of concepts they relate to the mathematical statements also seem to become more compartmentalized with fewer meaningful relations to other mathematical concepts; this includes the concept of function. The concept maps show that when the mathematical statement is recognized as a function it is understood to have fewer properties, and seems to be a less well-integrated concept as the preservice teachers level of performance in their mathematics studies decrease.

The preservice teachers less frequently consider different classes of functions in relation to the mathematical statements. None of the preservice teachers in the case study mention for example that \( y=x+5 \) and \( xy=2 \) corresponds to a linear or rational function, respectively, whereas the concept of quadratic function is evoked more frequently. Moreover, groupings or categorizations of functions regarding their different properties, for example properties they met during the calculus course such as continuous, monotone, differentiable, odd etc. is hardly recognized in the case study. Concepts related to the calculus course are rare in the preservice teachers description of the mathematical statements, with few exceptions like the concept of asymptote of a function.

The preservice teachers often make geometrical interpretations when describing functions’ different properties, but also numerical interpretations, in relation to the mathematical statements. In describing different properties of functions the preservice teachers less frequently use terminology related to the concept of function and mix terminology related to other mathematical concepts, with indication of root learning and less developed knowledge structures.

The form of the mathematical statements seems to influence what concepts the preservice teachers understand the mathematical statements to represent. It became clear that for example \( y-x=5 \), \( y-\pi x^2=0 \) and \( y=2/x \) are recognized as a diofantic equation, equation and function, respectively, and that an explicit form of the mathematical statements seems more frequently to be recognized as a function.

In the preservice teachers’ evoked concept images – related to the concept of function – there seem to be cognitive obstacles as well as prototypes. One cognitive obstacle seems to appear during the interview with the high achieving preservice teacher. Here her understanding and experience of the relations between the concept of an equation of a straight line and the concept of function seems to interfere with her reasoning about the relations between
the concept of equation and the concept of function. The preservice teachers in the case study gives the impression of to a lower extent have been in contact with problems that invites to deeper reflection upon the concept of function, functions’ different properties, and relations to other concepts.

5 Concluding discussion

The preservice teachers’ view of functions, contrasts with a view where the concept of function is a unifying concept in mathematics. This is clearly illustrated in the preservice teachers’ concept maps, where the concept of function rarely is a well integrated concept (as the preservice teachers’ less frequently notice relations between the concept of function and other concepts). The study indicates that the preservice teachers’ function concept is represented by a less developed knowledge structure; when the preservice teachers notice the concept of function, in relation to the mathematical statements in the study, they give less attention to various functional properties and infrequently associate functions with different categories of functions. Furthermore, when the preservice teachers do mention different properties of a function, it is often based upon a geometrical view, with elements of a numerical interpretation. In this context the preservice teachers less frequently use a terminology related to the concept of function, and frequently seem to lack a mathematical language to express themselves, and mix terminology related to other concepts than the function concept, which could be an expression of root learning (Ausubel, 1968, 2000; Ausubel et al., 1978).

A more developed conceptual knowledge of functions, as an object (Sfard, 1989, 1992) with a set of properties and an understanding of the function concept’s network of relations and significance in mathematics, is not prominent in the preservice teachers’ view upon functions as it appears in the study. This could mean that they as inservice teachers do not sufficiently promote the concept of function, or functional thinking, in their teaching. Furthermore, the concept maps give the impression that the preservice teachers’ knowledge structures tend to be compartmentalized, including the knowledge structures representing the concept of function, thus preventing preservice teachers from building a conceptual framework rich of meaningful connections. A consequence could be that the preservice teachers become less flexible in their way of reasoning about concepts, and are only able to offer a more limited range of teaching strategies regarding different concepts and their relations to functions, as inservice teachers.

The preservice teachers often seem to express an understanding of the function concept that is inconsistent with the definition. This became obvious in the case study. The case study also indicates that the preservice teachers to a lower extent have been in contact with tasks that invite to a deeper reflection
upon the concept of function, its different properties and relations to other mathematical concepts; thus not given the opportunity to deepen their understanding on these issues (Carpenter & Lehrer, 1999). One can argue (Cooney & Wilson, 1993; Eisenberg, 1992; Even, 1990, 1993, 1998; Thomas, 2003; Vollrath, 1994, to name a few) that it is important that preservice teachers have a more developed understanding of the function concept, noticing conceptual relations, properties, terminology and the concepts significance in mathematics. A conception of functions the preservice teachers in the study not yet seem to have achieved.

5.1 Implications for teaching

If one begins by assuming that preservice teachers’ view of the function concept is due to their experiences of functions and to a lesser extent the formal definition of the concept (Vinner, 1983, 1991, 1992; Vinner & Dreyfus, 1989), then a result of this should be that attention is focused on those functions which the preservice teachers encounter during a mathematics course. Highlighting both the mutual and distinct properties of different functions could be one way of stimulating a structural conception, as well as a means of contributing to the creation of more composite knowledge structures with respect to the function concept (Sfard, 1989, 1991, 1992; Slavit, 1997; Bagni, 2003). In this context, even a mathematical term for a property is of significance to the creation of better conditions to enable the property to become anchored to the cognitive structure and improve the preservice teachers’ ability to handle the concept and related terminology (Ausubel, 2000). Furthermore, the preservice teachers appear to have little experience of problem situations which allow them to reflect on the definition of the function concept. Stimulating a feedback of evoked concept images to the definition of the concept could be one way of enriching the preservice teachers’ concept image and develop their understanding of the function concept. This form of re-association to the definition of a concept occurs primarily with problems which are not of the standard variety, according to Vinner (1991, 1992). It appears that preservice teachers also need to be exposed to problem situations which concern the relation between the function concept and different mathematical concepts (Eisenberg, 1991; Vollrath, 1994), in an attempt to prevent the establishment of segmented knowledge structures and promote a flexible approach which would create better conditions for learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992). Reflection on the function concept, its relations to other concepts and its properties could be stimulated with the use of concept maps. One possibility is then to use concept maps in the unorthodox manner in which they were used in this study.
5.2 Comments on limitations of the methods in the study and scientific work criteria for quality of research

It is possible to criticize the study on several points. It could have involved a higher number of preservice teachers and been extended to teaching programs at several educational institutions. The study was conducted at a tertiary institution with relatively few preservice teachers of mathematics and sciences. As a result, e.g. for ethical reasons, it was not possible to reveal more detailed information about their background in connection to the case study in Paper III. However, the study is part of a research education program and should thus be conducted during a limited period of time, which ultimately affects the scope of the study. Various practical arrangements were also of significance to the execution of the study, such as having the opportunity to contact the preservice students, organizing the execution of the study, being in touch with the affected teachers and ensuring that the study could be conducted as planned. These would have influenced the choice of institution at which the study was to be conducted.

The preservice students answered questionnaires and drew concept maps during lecture time; this may have influenced the way in which they answered the questions and designed the concept maps. Arranging to have the preservice teachers draw the concept maps and answer the survey questions during lecture time ensured that the answers they gave and the maps they drew were independent and guaranteed the same conditions for each participant. This has significance in the comparison of their answers. Since I was the teacher during the lecture in which they drew the maps and the one who interviewed the preservice teachers, I could ensure that it proceeded as I had planned. The preservice students knew me as a teacher and this may have influenced their answers. I nevertheless found that it was good that the students who were interviewed knew me, since it facilitated natural interactions and I got the impression that it enabled them to express their opinions freely.

The interviews were based on the answers which the preservice teachers had given in the survey. The students were asked to comment on their answers. Nevertheless, in order to gain a better understanding of their thought processes, they were asked questions which may have influenced the manner in which they answered. The questions in the survey were open, which meant that it was necessary to ask additional questions in order to investigate more closely the preservice teachers’ perception of the function concept in relation to the statements. The interviews were often conducted during a predetermined time interval. This meant that it was sometimes necessary to limit the subject and move on to the next question. The study is based on voluntary participation and the time allotted to an interview (as described in section
4.2.3) was adapted in order to increase the number of preservice teachers who participated in the study. This was of value to the existing study and its planned continuation.

A number of criteria for quality of research have been promoted in mathematics education (Kilpatrick, 1993; Sierpinska, 1993; Lester & Lambdin, 1998, to name a few), where Lester and Lambdin (1998) suggests criteria such as openness, credibility, worthwhileness, coherence and competence, that I would like to use as a starting point in a discussion of the scientific quality and relevance of the study. I believe the study favors openness in describing how the study was conducted; describing relationships between researcher and subject, in appliance of methods, how data was collected and analyzed, with further samples of data given in the appendix. The research findings are grounded in multiple sources of evidence in form of questionnaires, interviews and concept maps, which should enhance credibility of the results in the study. The study is related to previous research in mathematics education and has gradually evolved during a period of time where earlier results have made an impact on the theoretical framework, research questions and methods used in the study. The gradual development of the study favor issues related to worthwhileness, coherence and competence of the research.

Among their research criterias, Lester and Lambdin (1998) find the criterion of worthwhileness the most important. To further comment this criteria, related research (in section two) show that teachers’ and preservice teachers’ conceptions of function internationally is a less well researched area – in contrast to students’ conceptions of function. Moreover, few studies have used concept maps in the study of individuals’ conceptions of function (and no previous study has to my knowledge used concept maps to investigate preservice teachers’ conceptions of function in relation to mathematical statements, as in the current study). Grevholm analysis results from a longitudinal study of preservice teachers’ conceptual development (Grevholm, 2000a, 2000b), including the concepts of equation and function. Few other studies have been conducted on preservice mathematics teachers in Sweden (Bergsten & Grevholm, 2004; Björkqvist, 2003), and to my knowledge, no other study has been conducted with a focus on preservice teachers’ conceptions of functions as in the current study.

5.3 Ethical aspects on the study

The identities of the preservice teachers are protected. All names used in this thesis are fictitious. Furthermore, the data in the study are not presented in such manner as to enable the identification of the participants. The preservice teachers have all been asked if they were willing to participate in the study. It is my impression that many of the preservice teachers who participated in the
study were able to benefit from the experience. This was particularly evident during the interviews, in which the preservice students commented on the interviews by revealing that the questions which were raised in the interview had helped them to improve their understanding of different concepts and relations. They also revealed that the interviews provided them with an opportunity to learn. This shows that their learning is influenced by our observations, which in turn influences the information that we measure; this is inevitable (Duit et al., 1996).

5.4 Future research

I plan to conduct further studies into preservice teachers’ view of the presence of the function concept in mathematics and in the mathematics school curriculum. I intend to study the significance of functions to preservice teachers and their perception of the relations between the function concept and other mathematical concepts. Studies into preservice teachers or inservice teachers ability to handle the concept of function in supervision should also be of interest. Questions about what different teaching strategies the teachers are able to offer and if they then sufficiently promote functions and functional thinking are relevant.

The study illustrates that preservice teachers’ views on the function concept rarely seem to be consistent with the definition of the concept. In this regard, I am interested in studying how different forms of supervision, problem formulations and problem solving methods can be utilized to develop preservice teachers’ understanding of the function concept. Attempting to promote feedback of evoked concept images to the concept definition, or by highlighting different properties of functions with the intention to form composite knowledge structures, are some ideas I have in this respect. Further, in the future I expect to conduct a deeper study of how technological tools can be applied to the situation.
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Paper I
Preservice teachers’ conceptions about \( y=x+5 \): Do they see a function?

Örjan Hansson and Barbro Grevholm

We are studying two groups of preservice teachers’ conceptions, progression and especially the concept of function in connection to \( y=x+5 \) when they are taking a course in algebra or one in calculus in their third and sixth term, respectively, in a teacher preparation program. There is a similar development in that they use to a higher degree a numerical interpretation before the course, which decreases after the course with a growth in linear and functional interpretation with the existence of two variables as a large and rather stable category. The group in their sixth term have a slightly more elaborated language and way of looking at \( y=x+5 \) than the group in the third term. For a majority of preservice teachers, in both groups, the concept of function is not evoked in connection to \( y=x+5 \).

1 Introduction

As parts of ongoing studies we have asked preservice teachers, in mathematics and science for school year 4-9, to answer the following question “We write \( y=x+5 \). What does that mean?” (Grevholm, 1998, 2002; Hansson, 2001). One reason to study \( y=x+5 \) is the fact that Blomhøj (1997) reported that final year students in compulsory school, age 15-16 years, have an unsatisfactory (see below) way of handling a question about how \( x \) is related to \( y \) in \( y=x+5 \). Another reason is that linear relations are common subjects that the preservice teachers are going to handle in different teaching-situations as inservice teachers. Linear relations are also often used in introductions of the function concept in later years of compulsory school.

The questions of the study are: What conceptions do preservice teachers have and what is their concept of function in connection to \( y=x+5 \)? What progression can be seen between two groups in their third and subsequently their sixth term, in a teacher preparation program?

2 Theoretical framework

Hiebert and Carpenter (1992) present a framework for examining issues of learning and teaching with understanding. The framework is based on the assumption that individuals’ knowledge is represented internally; that internal representations are structured and can be related or connected to one another to produce dynamic networks\(^1\) of knowledge. They suggest that we think

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\(^1\) The idea is supported by the fact that human memory, conceived as a network of entities, is a central and well founded theoretical construct in psychology and neuroscience (Anderson, 2000).
about these networks basically in terms of two metaphors, vertical hierarchy and web:

When networks are structured like hierarchies, some representations subsume other representations; representations fit as details underneath or within more general representations. Generalizations are examples of overarching or umbrella representations, whereas special cases are examples of details. ... a network can be structured like spider’s web. The junctures, or nodes, can be thought of as the pieces of represented information, and the threads between them as connections or relationships ... The webs may be very simple, resembling linear chains, or they may be extremely complex, with many connections emanating from each node. (Hiebert & Carpenter, 1992, p. 67)

The two metaphors can also be mixed, resulting in additional forms of networks.

The mathematics is understood if its mental representation is part of a network of representations. Understanding grows as the networks become larger and more organized “a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Existing networks influence the relationship that is constructed thereby helping to shape the new networks that are formed.

Some parts of the network are so tightly structured that they are accessed and applied as a whole, as a single chunk: “accessing any part of the chunk means accessing the entire network” (p. 75). Other parts, called schemata, are relatively stable internal networks that serve as templates to interpret specific events; that is abstract representations to which specific situations are connected as special cases.

Ausubel (2000) presents a hierarchical cognitive structure with similarities to the network model of Hiebert and Carpenter. He presents his theory for learning in an institutionalized setting and talks about meaningful learning and rote learning, which has consequences for the students’ cognitive structures. To accomplish meaningful learning for students teachers have to activate relevant “anchoring” ideas in the learners’ cognitive structures and it is necessary to build upon the learners’ prior knowledge; this is what Hiebert and Carpenter call the bottom-up approach (p. 81). When meaningful learning is accomplished then:

... eventually they [emergence of new meanings in semantic\(^2\) memory] become, sequentially and hierarchically, part of an organized system, related to other similar, topical organizations of ideas (knowledge) in cognitive structure. It is the eventual coalescence of many of these subsystems that constitutes or gives rise to a subject-matter discipline or a field of knowledge. Rote learning, on the other hand, obviously do not add to the substance or fabric of knowledge inasmuch as their relation to ex-

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\(^2\) Ausubel describes semantic memory as “Semantic memory is the ideational outcome of a meaningful (not rote) learning process as a result of which new meaning(s) emerge.” (p. x).
existing knowledge in cognitive structure is arbitrary, non-substantive, verbatim, peripheral, and generally of transient duration, utility, and significance. (Ausubel, 2000, p. x)

We consider what Ausubel calls meaningful learning to be similar to what Hiebert and Carpenter call learning with understanding where the dynamic network becomes larger and more organized with growing understanding; a similar phenomenon occurs in Ausubel’s model:

It is important to recognize that meaningful learning does not imply that new information forms a kind of simple bond with pre-existing elements of cognitive structure. On the contrary, only in rote learning does a simple arbitrary and non-substantive linkage occur with pre-existing cognitive structure. In meaningful learning the very process of acquiring information results in a modification of both the newly acquired information and of the specifically relevant aspect of cognitive structure to a specific relevant concept or proposition. (Ausubel, 2000, p. 3)

Tall and Vinner (1981) introduced the notion of concept image as “the concept image consists of all the cognitive structure in the individual’s mind that is associated with a given concept”, p. 151. Different parts of the concept image are evoked in different contexts and they say “we shall call the portion of the concept image which is activated at a particular time the evoked concept image”, p. 152. In this paper, we see a concept image as a chunk of the knowledge structure described above and an evoked concept image as a portion of the concept image in the way of Tall and Vinner.

### 3 Blomhøj’s study

Preservice teachers supervised by Blomhøj (1997) studied the concept of function in a group of 22 pupils that were in their final year of compulsory school (the 9th year). They asked the pupils to write their answers to the question “y= x+5, What can you say about x in relation to y?” and followed up the answers with interviews. In his report Blomhøj distributes the answers in four categories: a) answers that say that x is 5 less than y, b) answers that interpret the equation without answering the question, c) answers that say that x is 5 more than y and finally d) answers that neither interpret the equation nor answer the question.

The distribution of answers was that a) got 6, b) got 4, c) got 7 and finally the category d) got 5 answers. So category c), which is a wrong answer, includes the most answers. Moreover, the answers from the pupils often contain contradictions and more than half of the students could not give an acceptable interpretation.
4 Method and results

The preservice teachers at Kristianstad University are studying mathematics in their first, third and sixth term and are then taking courses of a total of 30 weeks full time study where approximately one third relates to educational studies in mathematics. We studied two separate groups of preservice teachers in their third and sixth term, respectively, of a four and a half-year teacher preparation program. The first one took place in the third term where Grevholm (1998, 2002) asked a group of 38 preservice teachers to answer a questionnaire that contained the question of interest before and after a five-week course in algebra and also interviewed some of the preservice teachers. The second took place in the sixth term where Hansson (2001) replicated the first study with a group of 19 preservice teachers in connection to a five-week course in calculus. Hansson also asked them to draw a map that represented their way of thinking about \( y=x+5 \) after the course.

Grevholm created a categorization based on the preservice teachers written answers to the question “We write \( y=x+5 \). What does that mean?” The categorization arose from the data that was gathered. The categories separate answers that:

1) describe how \( x \) and \( y \) are related numerically, here called category N
2) state that there are two variables, V
3) give a table of values for \( y=x+5 \), T
4) describe the relation as a straight line, L
5) describe the relation as a function, F
6) give other specific descriptions, O

Table 1 gives the distribution of answers. Hansson used the same categories and table 2 gives the distribution of answers. The tables are based on the total number of categories that the preservice teachers’ answers included.

<table>
<thead>
<tr>
<th>Category</th>
<th>N (46%)</th>
<th>V (27%)</th>
<th>T (5%)</th>
<th>L (7%)</th>
<th>F (10%)</th>
<th>O (5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>19</td>
<td>11</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>After</td>
<td>12</td>
<td>12</td>
<td>2</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1. 28 preservice teachers answered the question before and after the course in algebra; where 36 answered the questionnaire before and 31 after the course. Table from Grevholm (1998).

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\(^3\) The distribution of answers is adjusted compared to Hansson (2001) to become more uniform with the categorization of answers in Grevholm’s studies. In Hansson (2001) the categories F and V were broader and narrower respectively. Functional thinking (Vollrath, 1986) like “\( y \) depends on \( x \)” was graded F and more specific statements like “… variables \( x, y \)” graded V.
In the following we give an illustration of how the categorization was assessed (Fn and nF belong to the first and second study respectively):

N) F1: \( y = (\text{value of } x) + 5 \), 8M: \( y \) is a number that is 5 units larger than the number \( x \), F4: that \( y \) is the sum of 5 and the number you decide \( x \) to be.

V) F3: Two unknown, \( x \) and \( y \) are variables, 14M: \( y \) depends on \( x \), M10: Different for different people. For me it means that one \( x \)-value represents one \( y \)-value.

T) F7: A table of values with \( x \)-values on one line “\( x \) 5 4 3 2” and \( y \)-values on the second line “\( y \) 0 1 2 3”\(^4\).

L) M6: \( y = x + 5 \) is a line that intersects the \( y \)-axis when \( x = -5 \) and intersects the \( x \)-axis when \( y = 5 \), 12F: it can also be a straight line, 4F: You can also see it as an equation for a straight line that uniquely determines what the line looks like.

F) M5: Function \( y = \text{variable } x + \text{number } 5 \), 3F: \( y \) is a function of \( x \), 4F: \( y \) is a function of \( x+5 \).

O) 16F: It is also an example of an equation..., F7: It is an algebraic expression, M11: An equation with two unknown numbers.

Eight preservice teachers were also interviewed and tape-recorded in the first study and seven interviewed, four tape-recorded, in the second study. The interviews reveal that the students have more to say than they express in the answers of the questionnaire. In the conversation they usually give interpretations of “\( y = x + 5 \)” covering more of the categories N-O than in the questionnaire.

### 4.1 The use of concept maps

In the questionnaires students give only a few knowledge propositions to the question. Normally they give one and at most four propositions are given. In earlier research Grevholm (2000a, 2000b) has shown that the use of concept maps is one way to get students to reveal more about their mental representations. It is intellectually more demanding to draw a concept map than to answer a question. In the concept maps students activate more concepts and more links between them than in a verbal written proposition. Inspired by this experience Hansson also decided to use concept maps in his study.

\(^4\) In this case the preservice teacher puts \( x \) and \( y \) on the wrong values.
Map made by 5M. A concept map about \( y = x + 5 \).

The preservice teachers in the second study had some experience of drawing mind maps and concept maps in pedagogy and biology, so drawing maps was familiar to them. Hansson (2001) gave a lecture on how to use concept maps in mathematics education (in the way introduced by Novak & Gowin, 1984) and discussed how different maps can help visualization of knowledge and understanding and also be used as Ausubel’s advance organizers\(^5\). At the end of the lecture, he asked the preservice teachers to draw a map about \( y = x + 5 \) which they did for almost 30 minutes. One of the preservice teachers’ maps, made by 5M, is shown above.

5 Discussion

The two groups of preservice teachers we studied had a similar development of answers to the question before and after the courses they took in that there was a reduction in category N and growth in category L and F as shown in tables 1 and 2. Category V is large and quite stable in the algebra course; it is the largest category and rather stable in the calculus course. It is surprising that the preservice teachers in their sixth term before the calculus course came up with so many answers in category N; a category we judge as less advanced than the categories L and F. The ordering of categories N, V, T, L, F is a reflection of our view of order of more advanced levels of thinking; demanding more developed cognitive structures with the function concept having the most complex structure with connections to numerous sub concepts.

\(^5\) An advance organizer is a pedagogic device that helps … bridging the gap between what the learner already knows and what he needs to know if he is to learn new material most actively and expeditiously. (Ausubel, 2000, p. 11)
The answers in the second questionnaire were more detailed and covered more categories, and were more explicit in the group who took the calculus course. The authors could see a slightly more mature language in the answers of the second questionnaire and especially from the group who had advanced longer in the teacher preparation program, but a yet more elaborated language is desired to become successful as an inservice teacher. None of the groups used more advanced mathematics than from the curriculum of upper secondary level. The interviews indicate however that the understanding of the students is somewhat more developed than what seems to be the case in the written answers.

Examining data for individual students confirm that concepts develop slowly. More than every second student give answers in the same categories before and after the course. Other students just add an extra category. For those who keep the same categories it is notable that they after the course express themselves in a more advanced professional language for teachers. (The students were not aware of the categories that we use here. They were just asked to answer as honest and open as they could to show us their knowledge.)

There is a growth in the number of answers in category F when comparing the first and second questionnaire. But even so the majority of preservice teachers do not mention that \( y=x+5 \) is a function. Moreover, only one preservice teacher (M13) who took the algebra course gave \( y=x+5 \) any properties as a function. He wrote that it was a linear function and did so in the second questionnaire. There was also only one preservice teacher (4F) in the calculus course who, before the course, wrote that \( y=x+5 \) becomes a line. After the calculus course there were more answers mentioning that the function becomes a line (no one was referring to the line as a function graph); no other property was mentioned.

It is notable that so few preservice teachers write that \( y=x+5 \) is ‘an equation’ in the questionnaires. Only one preservice teacher mention it in the algebra course and four (in the second questionnaire) in the calculus course; in contrast to the concept maps where ten preservice teachers mention it. The concept of equation is one they have worked with for many years, much longer than the concept of function. It is also obvious that none of the groups is actively using the term linear. However, the concept of line is used in both groups and more frequently in connection to functions in the group studying in their sixth term.

When we look at the maps we see that they contain more information than the written answers. They are clearly more developed in the area of a straight line where they mention things like slope, intersection with the axis in a coordinate system and the equation of a straight line \( y=kx+m \); which was also visible in the written answers. Eight preservice teachers have function as a
part of their map, but it has few links\(^6\) connected to it (as in the map that 5M
drew). One exception is 17M who writes \(f(x)=x+5\) and gives the derivative
and primitive function. Moreover, links between function and straight line is
not common and only one map (14M) mentions graph and makes links be-
tween function, graph and then straight line. Other properties of functions like
for example monotony and continuity are not mentioned. The concept of
equation is more explicit in the maps than in the written answers and some
maps also have connections to applications and learning and teaching.

The fact that the group just took a course in calculus where the function
concept makes a central part was in large not visible in the written answers or
the maps they drew. It indicates that the function concept is less meaningful in
the context of \(y=x+5\). They seem to make connections with mathematical
knowledge on a less advanced level than what they have worked with in their
later courses in mathematics. A premature concept of function (Vollrath,
1986) is also visible in answer like “\(y\) depends on \(x\)” (14M) in category V.
Even category F has answers with a less developed concept of function like
“\(y\) is a function of \(x+5\)” (4F).

6 Conclusions

There was no indication that incorrect answers like those shown in the study
by Blomhøj were frequent among the preservice teachers. There is a similar
development in both groups of preservice teachers with tendencies of a nu-
merical interpretation of \(y=x+5\) before the course which lessens after the
courses with a growth of linear and functional interpretation; with the exis-
tence of two variables as a large and rather stable category. The group of
 preservice teachers who had progressed further in the teacher preparation pro-
gram had a slightly more developed language and flexible way of looking at
\(y=x+5\) where for example the concept of equation was more common. The
maps gave valuable information about how the students look upon \(y=x+5\) and
connections between different parts of knowledge became more explicit. The
function concept was not well developed in connection to \(y=x+5\); if men-
tioned it did not have any properties except as a line in a few cases. Views
upon the function concept, as an object with many properties, were hardly
visible. This became apparent in the written answers but also clearly in the
maps.

This study indicates that the preservice teachers’ concept of function is
not a rich cognitive structure in the evoked concept image in the context of
\(y=x+5\). It could mean that they as inservice teachers give less attention (Chin-
napppan & Thomas, 1999; Even, 1993; Fennema & Loef, 1992; Vollrath,

\(^6\) We assume the number of links is positively correlated to the concepts importance in relation to
each other in the context of \(y=x+5\).
1994) to the function concept in linear relations. The fact that teachers do not give enough explicit attention to the functional aspects of linear relations can be one explanation to the results from pupils in year nine in Blomhøj’s study. Niss (2001, p 43) concludes that “If it is something we want our pupils to know, understand or manage, we must make this part of an explicit and carefully designed teaching”. (Our translation). If we want students (pupils) to be able to interpret a given expression as a function this aspect must be part of the teaching that students are offered.

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References


Paper II
An unorthodox utilization of concept maps for mathematical statements: Preservice teachers’ response to a diagnostic tool

Örjan Hansson

This paper considers two groups of preservice teachers’ construction of concept maps in an unorthodox setting, starting with a mathematical statement. The purpose of drawing these maps is to investigate individuals’ knowledge and understanding about the statement; what concepts the statement represents, conceptions about these concepts and how they relate to each other. I describe how the preservice teachers respond to the activity of drawing such maps starting with the mathematical statements \(y = x + 5\) and \(y = \pi x^2\), respectively, with a focus on the concept of function. In the preservice teachers’ response there are signs of metacognitive activity and mediation in drawing the maps, but also indications of an ability to evoke conflicting cognitive pieces of information, where hierarchical maps seem more demanding to draw. The hierarchical maps also appear to have a higher potential as a diagnostic tool in the given context. Moreover, the maps indicate that different concepts – including the concept of function – related to the mathematical statements tend to be separated, preventing preservice teachers from building a conceptual framework rich of meaningful connections. Using evaluation principles influenced by Ausubel’s assimilation theory the utilization of concept maps seem to reveal important aspects of individuals’ knowledge and understanding in relation to the given mathematical statements.

1 Background

A concept map is a graphical representation of various connections between a number of concepts. The map consists of a network of nodes and links, where the nodes depict concepts and the links represent relations between the concepts. Concept maps were developed by Joseph Novak together with colleagues at Cornell University (Novak, 1990a, 1998; Novak & Gowin, 1984) and have been widely used in sciences education (Al-Kunifed & Wandersee, 1990; Novak, 1998; Novak & Wandersee, 1990). Similar maps have later been used within a range of different subject areas with varying theoretical frameworks (Ruiz-Primo & Shavelson, 1996), e.g., semantic networks (Fisher, 1990).

Concept maps have more recently been used in mathematics education (Doerr & Broers, 1999; Grevholm, 2000a, 2000b; Haseman & Mansfield, 1995; Leikin, Chazan & Yerushalmy, 2001; McGowen & Tall, 1999; Ryve, 2003; Williams, 1998, 2003), although to a limited extent; they have thus been formulated in many different ways. Concept maps, as presented by No-
vak, are based on Ausubel’s assimilation theory (Ausubel, 1968, 2000; Ausubel, Novak & Hanesian, 1978). This stipulates that the concept map be derived from a concept\(^1\) which is placed uppermost in a map which has a hierarchical structure, with general and inclusive concepts placed above more specific concepts.

Part of the data in Hansson (2003), and Hansson and Grevholm (2003) is derived from an unorthodox use of concept maps, in which the maps are drawn from a mathematical statement, rather than a specific concept. The concept maps were used to investigate preservice teachers’ comprehension of the mathematical statement \(y=x+5\) and in particular, the manner in which the function concept is derived. The investigations indicate that the maps contain a wealth of information on preservice teachers’ knowledge and understanding of the statement and the concepts it represents.

A mathematical statement (e.g. \(y=x^2\)) can be assumed to represent a number of different mathematical concepts (Goldin, 2002; Goldin & Kaput, 1996; Dreyfus, 1991) (where \(y=x^2\) may be an equation, a statement, function\(^2\) or a parabola, for example). Concept maps which are derived from a mathematical statement\(^3\) differ from those in which a mathematical concept is explicitly stated, since it could be interpreted as if the map was derived from the different concepts that the statement is understood to represent. The maps illustrate the concepts which an individual connects to the statement, the meaning and characteristics of the concepts and the relation between them with respect to the statement.

This paper is organized as follows: Aims and the theoretical framework of the study are followed by a section of methods and results from some of the few related studies in mathematics education. Thereafter the method and procedure for the present study is described as well as a framework for the analysis of concept maps. In the subsequent sections the results of the study are presented, including case studies of preservice students. The final section contains a concluding discussion.

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\(^1\) Concepts are by Novak defined as “perceived regularities in events or objects, or records of events or objects, designated by a label” (Novak, 1998, p. 21) and by Ausubel as “objects, events, situations, or properties that possess common criterial attributes and are designed by the same sign or symbol” (Ausubel, 2000, p. 2).

\(^2\) Let us assume that the statement is perceived to represent a real-valued function of a real variable.

\(^3\) I imply that a given mathematical statement (e.g. \(y=x^2\)) is not a concept according to the definitions given by Novak (1998) or Ausubel (2000). On the contrary, it can be perceived to represent a number of different mathematical concepts that can be illustrated in a concept map. Furthermore, if we use Novaks hierarchical concept maps and place a mathematical statement uppermost in the concept hierarchy then the map immediately assumes an unorthodox structure, since the hierarchy is lost. The statement is placed above the more general concepts that it is deemed to represent, i.e., where the statement constitutes a specific example for each of the concepts.
2 Aims of the research

The aim of this paper is to investigate and describe the use of concept maps to reveal preservice teachers knowledge and understanding of the concept of function in relation to the mathematical statements \( y=x+5 \) and \( y=\pi x^2 \).

More specifically, the study aims to answer the following questions: In what way do preservice teachers construct concept maps starting with the mathematical statements \( y=x+5 \) and \( y=\pi x^2 \) respectively? How is the concept of function expressed in the maps? What knowledge is displayed and what qualities are desirable in such a map? What experiences of drawing the maps do the preservice teachers express?

3 Theoretical framework

The framework is based on the assumption that knowledge is represented internally and understanding is described in terms of the way an individual’s mental representation is structured. Internal representations can be linked, metaphorically, forming dynamic networks\(^4\) of knowledge with different structures, especially in forms of webs and vertical hierarchies (Hiebert & Carpenter, 1992). The way in which an individual deals with or produces an external representation reveals something of how the individual has represented that information internally. Conversely, an external representation (a picture etc.) with which an individual interacts makes a difference in the way individuals represents the quantity or relationship internally.

Networks of knowledge are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information. According to Hiebert and Carpenter (1992) “a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Understanding grows as the networks become larger and more organized where existing networks influence relationships that is constructed thereby helping to shape the new networks that are formed. The construction of new relationships may force a reconfiguration of affected networks. Ultimately, understanding increases as the reorganizations yield more richly connected, cohesive networks, in learning with understanding. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. An individual’s ability to handle abstract notions, e.g. function, in a structural way (Sfard, 1991, 1992) as an object is in my view a reflection of more highly interconnected parts of the network.

\(^4\) The network model is a common model of semantic memory in cognitive psychology (Anderson, 2000) were some models, like Ausubel’s assimilation theory (Ausubel, 2000), makes further assumptions about the network like hierarchal etc.
Meaningful learning is a central idea in Ausubel’s assimilation theory (Ausubel, 1968, 2000; Ausubel et al., 1978). It is accomplished by using relevant anchoring ideas in the individual’s cognitive structure to obtain new meanings that “become, sequentially and hierarchically, part of an organized system, related to other similar, topical organizations of ideas (knowledge) in cognitive structure.” (Ausubel, 2000, p. x). Rote learning is the opposite to meaningful learning having a different impact on the cognitive structure in that “only in rote learning does a simple arbitrary and non-substantive linkage occur with pre-existing cognitive structure” (Ausubel, 2000, p. 3). New concepts are assimilated to the existing cognitive structure in the process of subsumption where the anchoring concepts are subsumers. Subsumers can be more elaborate and specific through integration with related concepts and new linkages can be established where the network of concepts will, in this way, be modified in the process of progressive differentiation. On real occasions when new concepts are introduced they can have a superordinate relationship to concepts that already exist in the cognitive structure, superordinate learning, which means that subordinate concepts acquire new meanings. When new ideas are integrated, the already existing concepts can recombine themselves and new meanings can be added to the existing concepts in what Ausubel calls integrative reconciliation. Assimilation theory applies what Hiebert and Carpenter (1992) describe as a bottom-up approach in the way knowledge structures develop, building upon the individual’s prior knowledge. I consider Ausubel’s meaningful learning similar to what Hiebert and Carpenter calls learning with understanding.

All parts of an individual’s cognitive structure that is associated with a given concept are called a concept image (Tall & Vinner, 1981). Different parts of the concept image are evoked at different times. The portion of the concept image that is evoked at a particular time is called an evoked concept image.

4 The use, definition and evaluation of concept maps in some recent studies in mathematics education related to the concept of function

There are a limited number of studies using concept maps in mathematics education, of which studies about the concept of function only constitutes a small part (e.g., Doerr & Browsers, 1999; Grevholm, 2000a, 2000b; Leikin et al., 2001; McGown & Tall, 1999; Williams, 1998). A more detailed account of studies related to the concept of function, their use of concept maps and the results are given in this section.
4.1 Calculus students’ view on the concept of function compared to professors with a Ph.D. in mathematics

In the study by Williams (1998) concept maps are examined as an instrument for assessment of conceptual understanding. The maps are used to compare the knowledge of function that students enrolled in university calculus classes’ hold. The students that participated in the study had different background and half of the students came from nontraditional\(^5\) sections and the other half from traditional sections of a first-year calculus course. Similar maps called experts’ maps were made by a number of professors with PhDs in mathematics.

The participants in the study drew their concept maps by hand starting with the concept of “function”. They were free to draw the maps as they pleased after an introduction of concept maps by Williams where she showed a number of examples of concept maps with different structure, such as hierarchical, non-hierarchical and web based concept maps, all with labeled links.

Williams states that the maps drawn by the participants proved to be widely divergent and did not lend themselves to a numerical scoring scheme. She takes a holistic approach to the evaluation process and describes her analysis of the maps as “I looked at the maps as integrated wholes and searched for differences between the two student groups and between the experts and the students.” (p. 416). In her analysis of the students maps Williams emphasis two observations. The first observation is that the concepts and propositions are trivial or irrelevant in many of the students’ concept maps, regarding both the reform and the traditional students. The second observation is that the students’ maps have an “algorithmic nature”, particularly among the traditional students, in that they reflect steps in a procedure. Another characteristic of the students’ maps is that integration of concepts, by linking a concept to a concept in another branch (cross-links), virtually did not exist. In contrast to the experts’ maps where cross-links were more frequent.

The experts’ maps, that Williams compared with the students’ maps, was a restricted task where the expert was asked to draw a concept map of function that represented what he or she would expect students completing the first-year calculus sequence to know. (Williams also asked the experts to draw an unrestricted concept map but she did not include them in the analysis.) These maps were more homogeneous than the students’ maps. Moreover, unlike many of the students’ maps, the experts’ maps showed no hint of algorithmic nature. Instead, they reflected properties, categorical groupings and classes of functions.

\(^5\) A reform section with emphasis on modeling and technology.
Williams concludes that the concept maps indicate differences in conceptual understanding. The general homogeneity of the experts’ maps and their distinct variance from the students’ maps lend in her opinion credibility to the conclusion that “concept maps do capture a representative sample of conceptual knowledge and can differentiate well among fairly disparate levels of understanding.” (p. 420). Moreover, in Williams’s experience the analysis of the maps also provided information about students’ understanding that is not readily gained from traditional pen-and-paper test. Providing important information about conceptual understanding and see it as useful in the mathematics education researchers’ collection of tools.

### 4.2 High achieving versus low achieving algebra students’ cognitive development in relation to the function concept

The study by McGown and Tall (1999) uses concept maps⁶ to document the processes by which students construct, organize and reconstruct their knowledge about functions during a course in mathematics. The students were taking a sixteen-week algebra course at a community college. They were asked to construct concept maps, starting with the concept of function, on three occasions at five-week intervals during the course. The students were advised to use Post-it notes to allow them to move items around before they drew their final maps, and at each occasion given an opportunity to review their map and redraw it one week later, without any given preference about the structure of the map.

Two groups of students, four of the most successful and four of the least successful students’ concept maps where analyzed (the selection of students was based on pre- and post-test, and the final exam). The major focus of the study was to trace the cognitive development of students throughout a mathematics course and to seek the qualitative differences between those of different levels of achievement. In addition to a qualitative analysis of the concept maps McGown and Tall also use a pictorial technique to document the changes in the maps in what they call “schematic diagrams”. A student’s schematic diagram illustrates how the student built the first concept map and the successive two maps by keeping some old elements, reorganizing and introducing new elements.

When the most successful students built their two successive concept maps, they added new elements to old elements in a structure that gradually increased in complexity and richness. In contrast, the least successful students

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⁶ It is worth noticing that McGown and Tall (1999) define a concept map as “A concept map is a diagram representing the conceptual structure of a subject discipline as a graph in which nodes represent concepts” (p. 281), and do not include labeled links in the concept maps in the study.
showed little constructive growth, building new maps on each occasion, and according to the authors:

Analysis of the … selected students reveals striking similarities among the schematic diagrams for each group. Each student in the more successful group produced a sequence of concept diagrams [concept maps] whose schematic diagrams retained the basic structure of the first within a growing cognitive collage. Each set of schematic diagrams for the least successful also exhibited a common characteristic: a new structure replaced the previous structure in each subsequent map, with few, if any elements of the previous map retained in the new structure. No basic structure was retained throughout. (McGown & Tall, 1999, p. 287, emphasis in original)

The low achieving students’ concept maps also reveals procedural undertones by concentrating on routines (find slope, find constant, solve, evaluate etc.), with few or no links to other branches of concepts. A quantitative analysis of the maps triangulated with the students’ written work and interview data confirms that their knowledge are compartmentalized, preventing them from building a conceptual framework with meaningful connections.

McGown and Tall conclude that there is a wide divergence in the quality of thinking processes, where high achievers show a level of flexible thinking in using various representational forms and building rich conceptual frameworks on anchoring concepts that develop in sophistication and power. Whereas the lower achievers reveal few stable concepts with conceptual frameworks that have few stable elements and leave the student with efforts to use learned routines in an inflexible and often inappropriate way.

4.3 Preservice teachers’ view on the function concept related to teaching and learning

In the study by Doerr and Browsers (1999) concept maps were seen as a tool for expressing preservice secondary mathematics teachers’ ideas about the concept of function. Concept maps were drawn individually before and after a course with several instructional sequences designed to challenge their existing knowledge about the concept of function and the mathematics of change, and to evoke models of how others might learn these ideas. The task was presented in an open-ended way and the preservice teachers were free to structure their concept maps in a way that seemed reasonable to them. They were also asked to write accompanying interpretative essays about the concept of function.

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7 The term “cognitive collage” is used to describe the notion of an individual’s conceptual framework in a given context.
The data analysis made by Doerr and Browers involve three phases, all in a qualitative manner. In the first phase overall structure and most central features were identified in the pre-course maps, followed by a second phase where these were compared to the preservice teachers’ essays and finally a third phase where the analysis focused explicitly on the differences between individuals’ pre- and post-course concept maps.

In their analysis, Dorr and Browers notice structural differences between the maps. The pre-course concept maps were structurally of two types. The first type of map was characterized by a central idea, usually “function” with a relatively large number of sub-concepts directly related to function, but with few details in the sub-concepts. These maps were often structurally similar to the central hub of a wheel. There was limited interconnectivity among the sub-concepts (few cross-links). The second type of map was in majority and more of an unstructured web with several clusters of sub-concepts linked to the concept of function. No preservice teacher had constructed a hierarchically organized map, flowing from top to bottom. This could according to the authors be in part a result from the web-based illustrations of concept maps that were given in class. (The authors do not give any more specific description of how they define a concept map, if they use labeled links etc. The authors comment that most of the preservice teachers had prior experience with concept maps and consider the preference for web-based maps, at least in part, a reflection of their thinking about the concept of function as non-hierarchical and non-linear.) The post-course concept maps showed shifts from the preservice teachers’ initial maps to more web-like structures with a higher degree of interconnectivity and a larger number of sub-concepts and more structured relationships among the sub-concepts.

Beside the maps structures, Doerr and Browers make some observations about the maps content. On their initial maps, all but one preservice teacher of the eleven preservice teachers that took part of the study included some ideas about multiple representations (table, graphs, symbols, words) of functions. For a large majority of the preservice teachers these representations were not linked to each other. On the final map, half of them now expressed a linked relationship between various representations of function. A major shift from the initial to the final maps was in relation to ideas about teaching and learning about functions. Only one preservice teacher had two single nodes in his initial map that were related to pedagogical strategies or student understandings. However, on the final maps, nine of the eleven teachers had added multiple nodes explicitly related to teaching about functions and learning issues for students. This suggests that prior to this particular course, preservice teachers’ views about the concept of function were largely disconnected from any pedagogical strategies or learning paths or obstacles that students might encounter. The preservice teachers’ final concept maps indicated that teaching
and learning issues were now linked to their mathematical understanding of the function concept, according to Doerr and Browers.

4.4 High school teachers’ understanding of the mathematics they teach

In the study by Leikin, Chazan and Yerushalmi (2001) high school teachers in mathematics were asked to draw concept maps for the concept “equation”. The purpose was to identify teachers’ conceptions of equation and how those conceptions fit inside a larger approach to school algebra, and especially whether the teachers hold a functions-based or an equations-based approach to school algebra. Moreover, they also wanted to examine the strengths and weakness of concept maps as a tool for examining teachers’ understanding of the mathematics that they teach.

The teachers made concept maps as a part of interviews at the beginning and at the end of a school year, in which they taught an algebra course that adopted a district-developed curriculum built around the assumption of technological support. The teachers were given a table of concepts and asked to draw a concept map, but was also allowed to add other concepts to the map. In case the interviewee did not know what a concept map was they were shown an example of a concept map for the notion of quadrilateral. (The authors do not specify how they define a concept map other than saying “In science education, concept maps – defined as a two-dimensional representation of relationships between selected concepts – have been used extensively…”, p. 289.)

At the end of the school year the teachers were asked to compare their new concept map with the map they drew at the beginning of the school year. The authors do not, in large, attempt to interpret or assess the maps other than make some qualitative comments about the maps’ structure, that contained web-like and hierarchical components. Instead, they saw the interviews as central to the evaluation of the maps and make the following comment whether the maps exposed the teachers’ understanding:

As a result, we would certainly advocate the importance of discussion of maps with interviewees, as well as the combination of concept maps with other interview tasks. Without such supports, it seems very difficult to assess what one sees and to determine whether repeated use of concept maps over time reveals changes in perspective or representations of different aspects of a teacher’s thinking. (Leikin et al., 2001, p. 295.)

This is a point of view somewhat different from other studies using concept maps in mathematics education, where they are highlighting the difficulty to assess concept maps. They also conclude that “One strength of concept maps is that they can indicate a conception of a concept as well as how this concept
fits into a larger web, potentially revealing tensions in a teachers’ thinking.” (p. 295).

Another outcome of the study was illustrated in two case studies that used concept maps in interviews, showing teachers’ difficulties in understanding relations between the concept of equation and the concept of function.

4.5 A longitudinal study of preservice students’ conceptual development

In a longitudinal study of preservice teachers’ conceptual development in mathematics, Grevholm (2000a, 2000b) use concept maps in a number of different ways:

   Concept maps are used as at tool both for analyzing the content of the teacher education to find the fundamental concepts, to investigate students’ answers in questionnaires and interviews and for the students to express their current concept structure. (Grevholm, 2000a, p. 1)

   In her description of concept maps Grevholm refers to Novak (1998) and defines a concept map in the same manner. Grevholm present findings of students’ conceptual development that illustrates the kind of analysis she does. In the longitudinal study, preservice teachers drew concept maps of the concept of equation and the concept of function at three occasions (Grevholm, 2000b) with about nine months respectively six months interval; the preservice students was not taking any mathematics courses during that period. The first concept map was often constructed in collaboration with other students while the second and third map were drawn individually. The preservice students participating in Grevholm’s study are in the same teacher preparation program as described in this paper.

   The concept maps were analyzed in a qualitative way to describe how they change over time. Grevholm (2000b) concludes “The knowledge about students’ conceptual structures given to me from the maps is much richer than the one I get from interview questions” (p. 16). Moreover, the results show that concept maps that students produce from time to time are similar to each other: “The slow development shown by the maps is towards clearer structure, richer maps and better verbal propositions. Maps from different students can be quite different but the knowledge a student expresses seems to be lasting with minor changes”. Grevholm also express her opinion that concept maps give valuable information of the students’ conceptual development “With the concept maps it is possible to see where changes in structures take place, how the individual concept images develop and where the learner still has some conceptual structure to explore and assimilate.” (p. 16). Another observation Grevholm makes in connection with the preservice teachers’ concept maps is
that the maps are useful to reveal when the students over-generalize and also when they not are able to group concepts that belong together.

5 Method and procedure

The present study is part of a larger study of preservice teachers’ understanding of the function concept, and was conducted during two consecutive spring terms. It comprises two groups of preservice teachers at a small university in Sweden and takes place at the end of each spring term, when the students have completed a course in calculus. At the end of the term the preservice teachers had completed all courses in mathematics in the teacher preparation program for mathematics and science, school grades 4 - 9.

In mathematics education there are several studies that are conducted with various types of maps, all of which are called “concept maps”. I have in agreement with the aims of the study chosen to investigate how the preservice students construct concept maps in a freely formulated form and in a hierarchical form, both of which are derived from a mathematical statement.

The first group of preservice teachers began by drawing a concept map for the statement $y=x+5$ structured in a way that seemed reasonable to them. One week later, they drew a hierarchical map for the same statement (the maps were collected at the end of both sessions). As a result, there was a lower degree of mutual influence between the contents of the maps and the student’s work methods. The preservice teachers’ maps were distributed shortly afterwards; they were able to give comments on the contents of the maps and how they experienced the drawing process.

Each concept map was analyzed as part of an integrated entity and elements such as structure, central features and included concepts were studied. The section of the map that contained the function concept (in case it was included on the map), its structure and scope, its relation to other concepts on the map and the characteristics that were highlighted with respect to it were especially considered. The different concepts that were thought to be represented by the statement and their relations to other concepts where also studied. The comments from the first group were compiled, together with the contents of the concept maps. (The same procedure was later repeated during the analysis of the concept maps which were drawn by the second group of preservice teachers).

After analyzing the concept maps prepared by the first group and their account of their experiences during the drawing process the work method for

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8 The university has 9,000 students, half of which are enrolled full-time.
9 The maps produced by the first group suggested that the freely formulated maps were often more extensive, but had a higher incidence of free associations (bringing mind maps to mind, Buzan, 1995) than the hierarchical maps. Among the nodes which were related to the statement, the number of concepts which the statement was understood to represent was generally lower in the freely
the second group was modified. The following term they drew both a freely formulated and a hierarchical map, but gave written comments on the maps on the same occasion. Like the preservice teachers in the previous group, the members of this group drew maps for the statement \( y=x+5 \); the study was then widened to include the statement \( y=\pi x^2 \).

The study was scheduled for an occasion of the preservice students’ mathematics course related to mathematics education. This resulted in lower absenteeism and ensured that the preservice teachers all had the same preconditions when drawing the concept maps, but then also limited the amount of time that could be allotted to the process of drawing the maps. The students were informed that the maps were intended to be part of a research project in which participation was voluntary.

### 5.1 The groups of preservice students which were studied

The preservice teachers that participated in the study were in the sixth term of a teacher training program in mathematics and sciences for the school grades 4-9. The training program’s duration was four and a half years. Applicants to the program should have successfully completed a program\(^{10}\) in mathematics and sciences at high school level, or be equally qualified. The two groups consisted of 19 and 25 students respectively, of which women were in predominant majority, who had studied 15 weeks of mathematics on a full-time basis during the term. On each occasion, the groups consisted of all preservice teachers in the sixth term of the program who were specializing in mathematics and sciences. At the end of the program they would have completed 30 weeks of full-time studies in mathematics, of which a third would have been dedicated to mathematics education.

### 5.2 Concept mapping

I gave a short presentation on concept maps to each student group and showed examples of different types of concept maps such as non-hierarchical, web-based maps and hierarchical maps, where the nodes represented concepts and formulated maps. Furthermore, relations between the concepts which were deemed to be represented by the statement – among them the function concept – tended to receive less attention in the freely formulated maps than in the hierarchical maps.

The evaluation of the first group indicated that the hierarchical maps had a greater potential to provide information about the preservice teachers’ function concepts and the relation between these and other concepts than the freely formulated maps had. The written comments and the author’s own observations of the students as they drew the maps revealed that the preservice teachers in the first group felt that the drawing of hierarchical maps was a more arduous task. As a result, the work method for the second group of preservice teachers was changed. The preservice teachers were now able to draw both maps at the same time and could thus manage the time as they chose.

\(^{10}\) Naturvetenskapligt program.
the links was labeled in each case (the hierarchical maps were constructed according to Novak & Gowin, 1984, and Novak, 1998). The maps which were displayed were often related to science education. Concept maps which were derived from mathematical concepts were largely avoided during the presentation so as not to influence the contents of the maps which the students were to draw in the subsequent assignment.

After the introduction, the preservice teachers in the first group were each directed to draw a concept map based on the statement \( y=x+5 \). They were thus able to construct a concept map with a structure that they themselves considered to be suitable. They were at the second time, one week later, asked to draw a hierarchical concept map for the statement \( y=x+5 \). The maps were distributed shortly thereafter and the students were instructed to examine their maps and give written comments on them. The instructions were as follows: “1. What are your thoughts on the two concept maps?” In particular, the preservice teachers were asked to comment on the different sections of the maps by answering the questions: “2. Is there any feature that you believe you have described successfully/unsuccesfully? Is there any feature about which you are uncertain?” and finally, against the background of their experiences of drawing the maps, “3. In your opinion, what are the advantages and disadvantages of concept maps?” The preservice teachers were also instructed to select, check and comment on a map drawn by another student within their group.

The second group of preservice teachers received a similar introduction to concept maps in the following spring term. They subsequently drew maps describing the statement \( y=x+5 \), one according to their own ideas and one which was hierarchically constructed. The procedure was repeated for the statement \( y=\pi x^2 \). Like the participants of the first group, the students were also given the opportunity to comment on their maps and their experiences of drawing the maps.

The first group of preservice teachers were given 30 minutes to draw their maps on each occasion, and somewhat more time to check and comment on their own maps and those of a fellow student. The other group received 60 - 80 minutes in which to draw and comment on their maps, depending on when the students decided they had completed assigned task. The author took about 30 minutes to introduce the maps to each group. 17 students from the first group and 24 from the other submitted all of their maps.

6 Framework for analysis

In the first place, I view the maps in this study as a qualitative tool to be used in the study of the individual’s knowledge and grasp of concepts with respect to a mathematical statement. Several studies on concept maps support the
theory that they provide valid information about an individual’s knowledge of a particular subject area, although most of the studies have been conducted on other areas than mathematics (Baralos, 2002; Laturno, 1994; Markham, Minterzes & Jones, 1994; Mintzes, Wandersee & Novak, 1999; Novak, 1998). The studies indicate that concept maps which are derived from a mathematical statement could perhaps be used as a diagnostic tool to qualitatively study an individual’s knowledge and interpretation of concepts (this is also indicated by Hansson, 2003; Hansson & Grevholm, 2003).

The nodes in the actual maps may be explicitly specified concepts or mathematical symbols, statements and expressions which are connected by links with linking words that explain their meaning. The mathematical statements, symbols and expressions invest the map with a symbolic language that enables associations with many different mathematical concepts (Goldin, 2002; Goldin & Kaput, 1996; Dreyfus, 1991).

6.1 Evaluation based on Ausubel’s assimilation theory

A hierarchical structure invites to an evaluation of the maps that, in part, is analogous with how Novak and his colleagues utilize Ausubel’s assimilation theory in their evaluation of concept maps (Ausubel, 1968, 2000; Ausubel et al., 1978). In using Ausubel’s theory, they concluded that concept maps should be hierarchical, starting with broad inclusive concepts that lead to more specific, less inclusive concepts, and that the maps should be labeled with appropriate linking words. Moreover:

Concept maps . . . are a representation of meaning . . . specific to a domain of knowledge, for a given context. We define concept as a perceived regularity in events or objects . . . designated by a label. . . . Two or more concepts can be linked together with words to form propositions and we see propositions as the units of psychological meaning. The meaning of any concept for a person would be represented by all the propositional linkages that the person could construct that include that concept. (Novak, 1990b, p. 29)

Concept maps were intended to “tap into a learner’s cognitive structure and to externalize, for both the learner and the teacher to see, what the learner already knows” (Novak & Gowin, 1984, p. 40). Novak and Gowin recognized that anyone representation would be incomplete – that not all concepts or propositions would be represented. Nevertheless, such maps would provide a “workable representation” (p. 40).

Ausubel’s assimilation theory suggests a hierarchical cognitive structure where new information often is relatable to and subsumable under more general, more inclusive concepts. The structure expands according to Ausubel’s principle of progressive differentiation where new concepts and new links are
added to the cognitive structure. Meaning increases for individuals as they recognize new links between different parts of the cognitive structure.

Using Ausubel’s assimilation theory Novak and Gowin (1984) suggests how to score their concept maps based on the following three main reasons:

1) More inclusive concepts and propositions are superordinate to less inclusive concepts and propositions. It takes a more active integration of concepts to construct a hierarchical concept. Individuals will have to grasp new meanings and actively integrate them into their existing conceptual framework. Representations of a hierarchical concept map thus show that an individual is able to differentiate a more inclusive concept from a less inclusive concept.

2) Concepts in cognitive structure continuously undergo changes of progressive differentiation; that is greater inclusiveness of concepts and propositions are acquired through linkages with other related concepts. As learning proceeds, more and more of related concepts are progressively differentiated. These progressive differentiations of concepts are enhanced when concept maps of one topic are cross-linked to concept maps for other related topic.

3) As more meanings are acquired, concepts are recognized and interconnected to other relevant concepts (in the process of progressive differentiation). Some newly acquired meanings can be conflicting to available concepts yet relatable. Integrative reconciliation occurs as new knowledge are learned and existing knowledge structures recombine themselves and new and different meanings arise, as when conflicting meanings of concepts are resolved or when misconception are uncovered. Concept maps that show valid cross-links between a number of concepts that otherwise be viewed as independent or conflicting can reveal learners’ integrative reconciliation of concepts.

6.2 Maps starting with a mathematical statement

The maps in this study differ from concept maps in that they are derived from a mathematical statement and not an explicitly expressed concept. An individual may believe that the mathematical statement represents several different concepts (Goldin, 2002; Goldin & Kaput, 1996) and that as such, the map is not derived from one concept but from several concepts. If the statement is thought to represent several different concepts then it is natural that they are directly linked to the statement (this is supported by the empirical results of the study). Thus like the maps which are recommended by Novak and his colleagues, these represent a meaningful, valid proposition (1 point), valid level in the hierarchy (5 points), cross-links (valid and significant 10 points, valid but do not illustrate a synthesis between sets of related concepts or propositions 2 points), examples (1 point). In addition, a criterion concept map may be constructed, and scored, for the material to be mapped, and the student scores divided by the criterion map score to give a percentage for comparison.

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11 A scoring procedure of concept maps is described in Novak and Gowin (1984) giving points for each meaningful, valid proposition (1 point), valid level in the hierarchy (5 points), cross-links (valid and significant 10 points, valid but do not illustrate a synthesis between sets of related concepts or propositions 2 points), examples (1 point). In addition, a criterion concept map may be constructed, and scored, for the material to be mapped, and the student scores divided by the criterion map score to give a percentage for comparison.
colleagues (see the figure below), ordinary concept maps can be drawn for each and every one of the concept and most of their evaluation principles could still be applied. Studies in mathematics education, as outlined in Williams (1998), and McGowen and Tall (1999) clearly indicate how differentiated, conceptual interpretations can arise in the maps. The various concepts which are represented by the mathematical statement can be analyzed individually, but the map also provides information on the individual’s interpretation of the relationship between the different concepts, in relation to the statement.

This is an example of a map, derived from the statement \( y=x^2 \), which represents the concepts of equation, formula, parabolas, …, function\(^\text{12} \) to an individual. It is possible to further extend the construction of the map by drawing ordinary concept maps for each of the concepts.

Here is another example of a map where the various concepts are arranged into a somewhat more integrated structure than in the previous map. The map contains information of the concepts, which an individual associates with the statement, the meaning assigned to the concepts that can be further developed to illustrate their relation to each other.

It can also be established that the current maps differ from hierarchical concept maps in that the mathematical statement violates the principle that gen-

\(^{12}\) Let us assume that the statement represents a real-valued function of a real variable.
eral, more inclusive concepts are placed above specific concepts when it is linked to the various concepts that it is thought to represent. The links from the statement violate the hierarchy since the statement constitutes a specific example of each of the different concepts. Nevertheless, the other parts of the map can be constructed hierarchically.

7 Results

7.1 The first group

The structure of the two maps which the preservice teachers drew differ greatly from each other. When the students drew their first concept map they were allowed to draw them in whichever manner they chose. They thus designed their maps in various different ways. Furthermore, they did not always write down words which explained the meaning of the links on the maps. On the second occasion, the students were instructed to draw the maps hierarchically, with the statement placed at the highest point on the map and asked to explain the links. It transpired that the students found it more mentally demanding to draw the maps in the latter manner. There were fewer concepts and fewer links to be drawn in the second map than in the first. One reason for this could be that the same amount of time was allotted to both sessions and that more time was required for the students to draw hierarchical maps.

A number of concepts were arranged around the statement on most of the freely formulated maps, such as the central part of a wheel (e.g., 4F, 5M, 12F). Other sections of the map consisted of a variety of web-like structures (including cross-links, even if they were in the minority) which could also resemble clusters (e.g. 3F, 7F, 13M). None of the freely formulated maps were hierarchical in structure; only a few of the maps had anything which resembled a hierarchical structure (e.g. 1F and 17M). Although the maps had numerous links, they did not give the impression that the preservice teachers had attempted to capture meaningful relationships between the concepts. As a result, clusters could be found in one section of the map, while other sections primarily consisted of chains of nodes and links (e.g., 10F and 7F).

The hierarchical maps that had been drawn by the students consisted of fewer links and nodes in general (with the exception of cross-links, in which case a distinct trend could not be identified). The structure of the maps became naturally more homogenous. The number of nodes which were connected to the statement \( y=x+5 \) was commonly reduced to fewer than half the number of links in comparison to the freely constructed map (e.g., 3F, 8M, 11M) where the statement in the extreme case had just one link (4F, 10F, 14M). As the number of links to the statement was reduced in the hierarchical maps, the proportion of concepts that the statement was deemed to represent increased and were thus allotted greater space on the maps.
The students tend to mix inclusive and specific concepts in the map. Students who are less successful at constructing their maps seem to focus on formulating sentences which they write down in the form of a chain, consisting of concepts and links (e.g., 6F, 7F, 13M), instead of focusing on constructing meaningful relationships between the different concepts in the map (this would result in fewer cross-links).

The maps also show signs of a less developed language among the preservice teachers, in that they do not always use appropriate mathematical terminology. For example, 7F states that an “equation” contains an “unknown variable”, 8M labels y=x+5 a “mathematical expression”\(^{13}\) and says that the line “slopes upwards”. 11M states that the solution to an “equation” may be “uniquely”\(^{14}\) defined (as opposed to “uniquely”\(^{15}\) defined), 12F writes that y=x+5 results in a “straight stoke”\(^{16}\) and that “the letters” (meaning the variables x and y) may be “different numbers”. The maps also contain traces of an algorithmic nature (in common with results in Williams, 1998, and McGown & Tall, 1999), which 5M raises in “the point-slope equation y-y_1=k(x-x_1)” in his map and 16F describes how the slope can be calculated “k=(y_2-y_1)/(x_2-x_1)”.

Of the various concepts which the preservice teachers connect to y=x+5, a graphical or geometric perspective in the form of a straight line occurs more frequently; in this case, the students grasp concepts such as sloping and intersection through the use of coordinate axes (e.g. 3F, 13M, 16F). In general, a more developed view of the function concept in the form of an object (Sfard, 1991, 1992) which possesses various properties (Slavit, 1997) and a set of sub-concepts (Eisenberg, 1991) will not usually be expressed in the maps. Eight students raise the function concept in both versions of their maps. While the students raise the function concept, none of them – from the first group – mentions that the statement represents a linear function. As opposed to those on the expert maps in Williams (1998) which represent various different classes of functions to a great extent. On the other hand, some students (14M and 17M) are increasingly able to reconnect to the function concept, even though this does not always take place when connecting to the node “function”, but instead connect to other nodes on the map (often to “y=x+5”), where 17M raises the “derivative”, “the anti-derivative” and “the inverse”. (Incidentally, 17M is the only student in the group who uses the symbol f(x)=x+5 in connection to the function concept.) None of the students mentioned the domains or codomains in the maps which contained the function concept.

\(^{13}\) According to Swedish terminology, a mathematical expression does not contain an equality sign.
\(^{14}\) “Mångtydigt” (swe.).
\(^{15}\) “Entydigt” (swe.).
\(^{16}\) “Rakt streck” (swe.).
One characteristic of the preservice teachers’ maps was that when the function concept was actually present, it was not a well-integrated node with connections to several nodes on the map. The preservice teachers do not observe the influence of the other concepts on the function concept. For example, only one student (14M) who had the terms “straight line” and “function” in his map states that the line is a graph of the function. Furthermore, in those cases where the preservice teachers included the concepts “function” and “increasing” (e.g. 2M, who uses the term “increasing” to refer to the slope of the line), they did not link “function” and “increasing” and do not seem to grasp the fact that “increasing” is also a property of the function. Another example of this is shown by 7F, who states in her (freely formulated) map that the x-value -5 results in a “point of intersection” on the x axis, but does not raise the fact that -5 is zero of the “function” shown on her map. The examples all represent an absence of cross-links, with respect to the function concept.

Although the preservice teachers’ two maps have completely different structures (since the second map has a hierarchical structure while the first does not) and the second map has fewer nodes and links, common “themes” such as straight lines, equations and functions are continuously repeated as a rule. Connections to teaching and learning rarely appear in the maps. Only a few preservice teachers (such as 4F and 11M) make this connection to students’ learning; even so, this does not constitute a well-developed area, but consists of one or a few nodes on the map.

7.1.1 Preservice teachers’ response on drawing the maps

The students in the group constructed their hierarchical maps with varying degrees of success. Their comments on the maps reveal that several of them prefer to use maps as a tool during brainstorming sessions, in which they write down their thoughts on the statement, connecting them with links as they to their understanding did when they drew the first map. Several students indicated that they found this work method to be more natural; for example, 9F wrote the following comment: “I think that it is more difficult to make a hierarchical concept map than the first type of map we did. When I constructed the first map I could follow my own ideas and work in accordance with my own sense of logic. In the hierarchical map I have to ‘rearrange’ my thoughts.” Similarly, 12F wrote: “I could consider using this during a brainstorming session with other students during which they write down the ideas they come up with and what they think about. However, this would result in a map which is much more like the first. To draw the second map one has to know what one is doing in order to be able to proceed”.

The lack of experience of drawing maps is also reflected in the students’ comments. Many of them feel that the maps are “confusing”, as can be seen
from 6F’s comment, “they can quickly become confusing”, or 2M’s comment, “it can be difficult to arrange the concepts in order of hierarchy, but it becomes apparent that the hierarchical order is easier to understand after attempts are made to do so. The latter [hierarchical map] has an order and a structure, while the former [non-hierarchical map] has concepts all over the place.” (It can also be mentioned that Novak & Gowin, 1984, and Novak, 1998, state that students must practice drawing hierarchical maps so as to be able to utilize them more successfully. Although their comments refer to (orthodox) concept maps, their experiences may be relevant even if the maps are used in a more unorthodox manner).

The students’ comments on their colleagues’ maps were often brief. They generally thought that the hierarchical map was easier to follow and understand, as evidenced by 4F’s comments on 2M’s map: “the second (map) is clearer and more easy to follow, from start to end”, or 12F’s comments on the map which was drawn by 15F: “the first map is more confusing than the second, but one cannot say whether it is good or bad since concept maps are so individualistic. It is as if one were to say that notes are also individualistic, thus the person who wrote them may understand them, but another reader may not need to understand.” At this point, 12F refers to a view that was also raised in the comments, in which the students did not appreciate maps as a tool for the assessment of knowledge and understanding, but rather as very individualistic constructions of value to the creators themselves, but not to other. Their comments reveal a perception of concept maps as a metacognitive tool (Novak, 1990b).

In general, the students are more expansive when they comment on their own maps. They also go as far as to establish connections to the maps’ structure and the suggested work methods. The first map was considered to be “confusing”, but arose from the work method in which participants were invited to write down their thoughts and associations; many students preferred this method. The second map was considered simpler to follow and easier to interpret, but more difficult and demanding to draw.

The answers to the questions which are connected to the maps’ layout vary, but are nevertheless often mainly connected to the first task, where the structure of the maps is affected and includes “confusing” maps as well as a lack of experience in drawing maps. Among the common answers to the question of how the students from their experiences felt about working with maps, is that while the maps can be useful in obtaining a clear image or a comprehensive impression of how an individual interpret the mathematical statement, they can also quickly become unclear and “confusing”. There are also answers which concern metacognitive aspects (Novak, 1985, 1990b); as 2M writes, “good tool to help a person to gather his/her thoughts and focus his/her skills, or 9F: “It helps a person to structure his/her thoughts.”.
7.1.2 Two preservice teachers’ maps

The layouts of maps drawn by two students Ted (5M) and Ann (7F) and their accompanying comments are presented below. Both preservice teachers raise the function concept in their maps. The maps are examples of how the various students in the group drew their maps and illustrate that they are very individualistic in nature.

In their freely constructed maps, both of the preservice teachers placed the statement in the center, as did most of the students in the group. Nevertheless, they are closer to the group’s extremes with respect to the maps’ web-like structure and trend towards clusters, which are more predominant in Ann’s map than in Ted’s. Ted’s hierarchical map is also more representative of the manner in which the students designed their own hierarchical maps, while Ann’s map is an extreme case with respect to the absence of cross-links.

7.1.2.1 Ted’s maps and comments

The first map that Ted drew.
The second map that Ted drew.

<table>
<thead>
<tr>
<th>Found only in the first map</th>
<th>Found in both maps</th>
<th>Found only in the second map</th>
</tr>
</thead>
<tbody>
<tr>
<td>motion, distance, acceleration can describe a physical event, graphical solution, variable, infinite, calculator, intersects line 5 above, y is 5 units larger than x, general form.</td>
<td>straight line, coordinate system, origin, y=kx+m, y depends on x, a slope of 1, function, table of values.</td>
<td>y-y1=k(x-x1), y-axis 5 steps above the origin, at the point (0,5), mathematical expression, an x-value returns one y-value only. (Links: point-slope equation, describe, definition, intersect, direction, graphically shown in, values can be described in.)</td>
</tr>
</tbody>
</table>

Overview of the contents of the maps (the description of the links’ meaning has also been included in the other map).

Ted’s answers: 1) More order in the second map. More “objects” in the first. Would nevertheless use the second map in a teaching scenario. 2) I have managed to give a good description of the straight line [this is in reference to the second map]. The first map is confusing. 3) Overview. Some people may find it confusing.

The first map resembles a mind map (Buzan, 1995), where the statement y=x+5 is located in the center of the map, from which links (12) are drawn to the various concepts and assertions which Ted makes in connection to the statement. Hardly any attempt is made to place the concepts on the map in order of hierarchy, or to arrange them in a more systematic manner. The structure is similar to one of the types of pre-course maps, “the central hub of a wheel”, described in Doerr and Brower (1999). The meanings of the vari-
ous links are not explained in the map. The map simply branches off at the statement “can describe a physical event”, which directed has three links indicating that it represents a more progressively differentiated structure of knowledge (Ausubel, 1968, 2000; Ausubel et al., 1978). Other links do not branch off, with the exception of “straight line,” and different sections of the map are not linked; the map does not have any cross-links. It is observed that a fundamental concept such as “function” is not connected to many sections of the map. This could be interpreted as an indication that the statement does not evoke a rich concept image of function (Vinner, 1983, 1992; Vinner & Dreyfus, 1989), and thus that the concept of function is represented by a less developed knowledge structure with few relations to other concepts (Hiebert & Carpenter, 1992).

The second map is, to a great extent, constructed hierarchically and contains more inclusive concepts that are placed above specific concepts, with no descriptions on some of the links. The structure of the map is completely different from that of the previous map. Even the contents have changed (see the above table). It is observed that Ted uses three terms when the concepts are to be arranged in hierarchical order: “straight line”, “function” and “mathematical expression”. Links to these concepts are drawn from the statement. It can be said that the three expressions are the most representative of the concepts in relation to the statement. The placement of the first two concepts close to the statement seems to indicate that they are more prominent. “Straight line” is the most developed of the three concepts, since it has the most links and thus gives rise to the majority of the detailed concepts and statements on the map. It can be noted that there are no links between the three concepts or their underlying sections on the map, which indicates a lower degree of progressive differentiation and that this is particularly apparent for the concepts “function” and “mathematical expression”. It can further be noted that there is a potential need of integrative reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978), e.g. in the absence of links between “y depends on x” and the region related to the function concept in the map. A less developed view of the concept “mathematical expression” is confirmed in this case by that the concept have not been further developed on the map and the fact that y=x+5 do not represent a mathematical expression. “Coordinate system” is the term which has the most links, indicating that it is one of the more conceptually developed notions and (in Ted’s view) one of the most meaningful concepts in the map (Hiebert & Carpenter, 1992).

If the function concept is studied in detail in the same manner as with the previous map then the link labeled “definition” from “function” to “an x value returns one y value only” indicates an understanding of function as a process (Eisenberg, 1991; Sfard, 1991, 1992). The link labeled “definition” to the node is intended to be viewed as a definition for the function concept and Ted
notice the univalence criteria “an x value returns one y value only” but other components, as domain and codomain are not raised. Although the function concept is a concept which (via a link from y=x+5) is considered to represent or “describe” the statement and a more prominent concept (located on the row closest to the statement) on the map, it does not result in a more developed sub-structure on the map. Neither is it linked to a representational structure such as a graph, table of values or any characteristic of y=x+5 as a function (Chinnappan & Thomas, 2001; Even, 1998; Thomas, 2003). The evoked conceptual images which Ted has of function do not lead him to connect the concept to other nodes on the map. Together, they indicate that the function concept has a less developed knowledge structure, particularly for linear functions. A comparison with the previous map reveals that more information on Ted’s conceptions of the function concept appears in his second map than in the first. The previous map may give the impression that Ted did not view the function as a more representative concept than the other 11 concepts which were linked to the statement y=x+5. The second map seems to reveal that this is not the case, and for Ted a highly representative concept. Nevertheless, it is less developed and has few properties since it lacks an underlying section of the map which could accommodate concepts such as “linear”, “increasing” etc.

Ted has constructed the two maps in completely different ways. His comments reveal that he identifies the structure of the maps and thus describes the second map as having “more structure”. Ted states that of the two maps, he would be more likely to use the second in a teaching scenario; he also feels that he has succeeded in giving a good description of the straight line. (It can be noted here that the preservice student uses one of the three concepts which are directly linked to the statement y=x+5 to conclude that the lower section of the map includes the concept “straight line”). He describes the first map as “confusing”, but states that it contains more “objects”. Although Ted uses the term “object”, this is a conception which is not reflected in all sections of the maps; this includes those sections where the definition of “function” suggests a process conception (Eisenberg, 1991; Sfard, 1992; Tall, 1992, 1996). This view is related to procedural understanding (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986) which is prominent in studies conducted by Williams (1998), and McGown and Tall (1999). He also raises the term “overall impression” as an advantage of the type of maps which the preservice teachers have drawn; after comparing the maps he states that a “confusing” map, i.e., an unstructured map is not to his preference.
7.1.2.2 Ann’s maps and comments

The first map that Ann drew.

The second map that Ann drew.

<table>
<thead>
<tr>
<th>Found only in the first map</th>
<th>Found in both maps</th>
<th>Found only in the second map</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 y-axis, -5 x-axis, point of intersection, slope, unknown variable, equation, the slope of the line, line, mathematical expression</td>
<td>Coordinate system, x-axis, y-axis, y=kx+m, the equation of the line, dependence, function</td>
<td>variables, slope of a line, describe the intersection on the y axis, coordinates, a curve, straight line, special case curves</td>
</tr>
</tbody>
</table>

Overview of map contents.
Ann’s answers: 1) I think that they are confusing and difficult to comprehend. The non-hierarchical map is nevertheless a bit better, since it has more features which are connected in various ways. 2) I think that I have managed to give a relatively good description of the line’s equation. I think that the structure of the map is poor and confusing. 3) I think that they are confusing and difficult to visualize; however, they provide students with a new way of learning by using an image instead of simply using text. In other words, a more aesthetic approach to learning.

In the first map that Ann drew, the statement y=x+5 is surrounded by six different nodes representing concepts and a statement to which directed links are drawn from the main statement, i.e., y=x+5. To a certain extent, the map has a web-like structure and is more developed around the nodes “y=kx+m”, “coordinate system” and “point of intersection”, where the first node has six connections and the latter two have five connections to other nodes on the map (a two-way link is counted as one connection). This indicates that these are more developed concepts.

It may be natural to insert more links between the nodes on the map, e.g. between “line” and “slope” or “line” and “the equation of the line”. This could indicate a limited, progressive differentiation (Ausubel, 1968, 2000; Ausubel et al., 1978) of the related concepts. Furthermore, an expression such as “function” may be considered to be peripheral in relation to the statement, since it is only connected to one other node on the map. Similarly, the term “line” has only one link and the contrast with the statement “y=kx+m”, for example, is remarkable. Moreover, the expression “unknown variable” is found on the map (with a link to “equation”), clearly indicating a less developed variable concept (which is also confirmed by the fact that the concept has just a single link on the map). No clear attempt to structure the contents of the map has been identified.

When Ann draws a hierarchically structured map she uses four expressions: “function”, “the equation of a line”, “coordinates” and “a curve”. They are connected to the statement “y=x+5” which is located at the same level and can be viewed as the most prominently represented concepts (Goldin, 2002; Goldin & Kaput, 1996) with reference to the statement. As in the maps produced by most of the students in the group, the number of links has been reduced in the hierarchical map, including the number of links from the statement y=x+5.

The map’s structure is dominated by the four most prominently represented concepts, each of which are linked to underlying concepts and statements through separate chains which are not connected to each other. It is
obvious that the task of arranging concepts in a hierarchical structure leads
Ann to focus on four expressions which are each developed separately. There
are significant contrasts between this and the first map. Many concepts are
now connected in a different sequence than that of the previous map. In the
second map, Ann has thoroughly linked the nodes in chains; she has even
gone so far as to construct a chain (with respect to the statement “y=kx+m”) where the links “where k” and “and m” follow each other, instead of allowing
both links be derived from “y=kx+m”.

A comparison of the two maps drawn by Ann reveals that she had drawn
many more links between the nodes on the first map than on the second. The
second map has mainly developed from concepts which resulted in separate
chains of associations. This indicates that Ann has devoted too short a time to reflect upon the included concepts and their relations when drawing the map
to be able to use it as a diagnostic tool.

If the function concept is studied in detail then it is observed that it recurs
in the hierarchical map as a concept which is directly linked to y=x+5 and is
thus one of the statement’s most prominently represented concepts. Upon
comparing this to the first map, Ann changes the order between “function”
and “dependence”. It is observed that the function concept on the hierarchical
map is viewed as being a central concept in relation to the statement y=x+5,
in contrast to the case with the freely formulated map, but is still a less de-
veloped concept.

The comments reveal that Ann thinks that the maps are “confusing and difficult to
interpret”. She states that the first map they drew was the better of the two since it con-
tained “more information” and “is interconnected in a different way”. Ann implies that
“the equation of the line” is something that she has “been relatively successful at describ-
ing”; it can also be stated that the first map is more developed in the equation of a straight
line environment, as well as that “the equation of the line” constitutes the longest chain on
the second map. The comments reveal that Ann perceives the maps to be of minor im-
portance. This may even bring some bearing on the layout of the second map, since it reflects
few meaningful relations and limited conceptual understanding (Hiebert & Carpenter,
1992). One can observe that Ann also makes connections to learning in her comments
when she states that the maps give rise to “a different style of learning” and discusses the
term “image” in relation to “text”.

7.2 The second group
After having evaluated the maps which were drawn by the first group, it was
observed that while the freely formulated maps were often more comprehen-
sive, they nevertheless had a higher component of free associations (bringing
“mind maps” to mind) than the hierarchical maps. The proportion of nodes

few cross-links on the maps; this trend is also found on the maps which were drawn by students in
studies conducted by Williams (1998), and Doerr and Browsers (1999).
which the statement was perceived to represent was generally lower, and meaningful relations between these nodes tended to receive less attention than those in the hierarchical maps. The students also perceived the hierarchical maps to be more difficult to draw, which resulted in a modified work method in the second group.

The preservice teachers in the other group drew their maps during a longer, continuous period; they were asked to draw and comment on two maps for the statements \( y = x + 5 \) and \( y = \pi x^2 \) respectively. One map was to be freely designed, while the other was to have a hierarchical format. It was suggested that the students begin by drawing the free-format map, after which they should draw the hierarchical map. One of the consequences was that the students often based their hierarchical maps on their free-format map. As a result, the hierarchical maps generally became more comprehensive than the freely formulated maps. Most students added supplemented the second map with additional concepts and statements since they were drawing on the basis of their freely formulated map.

### 7.2.1 One preservice teacher’s maps

Eva’s (F16) map below can be used to illustrate how the maps which the students drew were constructed, as well as to trace the manner in which she designed the layout and commented on her maps. Her work method corresponds with the general format adopted by the students in the group.

#### 7.2.1.1 Eva’s maps and comments

The first map that Eva drew for \( y = x + 5 \).
The first map that Eva drew for \( y = \pi x^2 \).

**Eva’s comments on the map for \( y = x + 5 \):**

Thoughts on this map: Difficult to get started. A little bit “fixed”. Happiest with the function section. The inverse section is rather poor.

**Eva’s comments on the map for \( y = \pi x^2 \):**

Would have been able to write much more on this one. Easy to form associations to it. Personally, I rarely use concept maps but I think that it is an excellent idea to gather and present my thoughts (to myself). I think that the “function” and “\( \pi \)” sections are ok. This is something which needs training, just like other work methods. Good selection of supporting terminology and posts, e.g. in a accounting process. Maps are useful for presenting an overall view of a given area.

The second map that Eva drew for \( y = x + 5 \).
The second map that Eva drew for $y=\pi x^2$.

_Eva's comments on the two hierarchical maps:_

Thoughts on these: Clarify by using a hierarchical order. Tried to make it clearer with color. Difficult to make it look good. Perhaps it should be constructed more like a family tree. Reworking the origin is a positive move. Need to think about it one more time.

It is clear that Eva made use of information from her first maps, which were freely formatted, when she drew the hierarchical maps. If one contemplates the two maps for $y=x+5$ then it becomes evident that while she has extended the hierarchical map with “$y$ depends on $x$”, she has kept the nodes and links from the first map. There are, however, some small variations in her choice of words. The biggest deviation is “function of $x$”, which was later changed to “function” when she added the node “$y$ depends on $x$”.

On the hierarchical map, the four concepts “mathematical formula”, “equation”, “inverse” and “function” were directly linked to the statement $y=x+5$ and can be seen as strongly represented concepts (Goldin, 2002; Goldin & Kaput, 1996) with respect to the statement. In addition, “function” has the most links, which indicates that it is the most developed concept of the four, and may be confirmed by the fact that Eva writes that she is “happiest with the function section” in her comments. The four concepts and their underlying structures are not linked on the map, despite the fact that there is room for several cross-links. This implies a need for progressive differentiation and integrative reconciliation (Novak, 1998; Novak & Gowin, 1984).

If we specifically study the function concept in relation to $y=x+5$, then we realize that Eva ties the variable concept and a dependent variable to

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18 The student colored the nodes in order to highlight the level to which they belonged in the hierarchy.
“function” and applies a geometric interpretation with respect to the function concept. But she does not use specific terminology such as “graph” (she writes “give graphically” instead) or describe properties (Slavit, 1997; Williams, 1998) such as “linear”, “increasing” and “continuous”, which apparently do not exist in her evoked concept image of “function” (Vinner, 1983, 1992; Vinner & Dreyfus, 1989) in the context of the statement. Neither does she mention domain and codomain. Furthermore, she has directly linked “inverse” to the statement $y=x+5$ without making any connection to the function concept. Her comment that “the inverse section is rather poor” indicates that the map invites contemplation of the term “inverse”, which may benefit integrative reconciliation (Novak, 1998; Novak & Gowin, 1984).

There are a number of differences between the nodes and links in the two maps which Eva drew for the statement $y=\pi x^2$. In the second map, the terms “equation” and “unknown” have disappeared, while “inverse” and “$x=\sqrt[2]{\frac{y}{\pi}}$” have appeared. Furthermore, “$\pi$” has been expanded to become “Greek letter” and “infinite number of decimals” and “3.14”. The link to “min” (previously labeled “min value”) has also been changed. “$\pi$” has the largest number of links on both maps (followed by “function”) and apparently belong to one of her more progressively differentiated knowledge structures related to the content on the maps (Ausubel, 1968, 2000; Ausubel et al., 1978; Novak, 1998).

Eva comments on her first map for $y=\pi x^2$, drawn in freely-chosen format, by stating that it is “easy to associate (things) to it”, thus indicating that she views the construction of the map as a process with associations to various concepts, expressions and so on. This is the same with the opinions put forward by a majority of the students (e.g. M8 and F12) while they constructed their freely formatted maps. The other map has been changed to a certain extent; Eva’s comments reveal that she believes that the hierarchical format makes the map “clearer” (although the map is not adapted to a hierarchical structure, considering the concepts). The three terms “$\pi$”, “inverse” and “function” are directly linked to the statement and may be perceived as the most strongly associated concepts in relation to $y=\pi x^\frac{2}{2}$. “$\pi$” and “function” have the highest number of links of the three terms, which indicates that they are represented by more progressively developed knowledge structures. These are the concepts which Eva believes to “ok”, as she writes in her comments. She also writes that she can develop the two further; this takes place in the hierarchical map. There is a large contrast between “inverse” and the two other concepts which are connected to $y=\pi x^2$, since these both give rise to more comprehensive underlying structures, while “inverse” is only linked to one underlying node. The number of links which are connected to “inverse” indicates a low degree of progressive differentiation (Ausbel, 1968, 2000;
Ausubel et al., 1978) of the related cognitive structure and what seems a low degree of comprehension (Hiebert & Carpenter, 1992). This is verified by the fact that \( y=\pi x^2 \) has no inverse (assuming that the statement represents a real-valued function of a real variable) and that there is no link between “inverse” and “function”. Concerning the function concept, it can be noted that Eva draws a parallel to a function class “2nd degree function” of polynomial functions – as opposed to \( y=x+5 \), where no function class was specified – and highlights characteristics such as “max” and “min” (even though they are tied to “parabola” and there are no links to “2nd degree function”).

Parts of the two hierarchical maps are similar in structure for the two mathematical expressions (primarily those sections surrounding “inverse” and “function”). This may be due to the fact that the maps were subject to mutual influences during construction, in being constructed at the same occasion. Similar tendencies are found among other students (e.g. F1, F2, M8) in the group.

The written comments indicate that Eva believes that the maps provide a good overview or “an overall view of a given area” and appears to view this primarily as a metacognitive tool (Novak, 1985, 1990b), commenting that it is “an excellent way to assemble and present my ideas (to myself)”. She also comments on the work method which was applied during the construction of her first map, “reworking the origin” describing it as “a positive move” and state that she need to think one more time. This reflects the general comments which many students in the group (e.g. F2, F5, F8) gave on the work method, although there were students who preferred to draw the hierarchical map after having first written a list of concepts (e.g. M7).

7.2.2 Preservice teachers’ response to the drawing of the maps

In addition to Eva’s comments, there were other comments from the group related to metacognition (e.g. F9, F11, F14), in addition to comments which indicated that the student perceived the maps to have a mediating role, e.g. M6, who states that “you see relations that you have not considered”, F3, who writes that “one really has to consider the meaning of what the different things mean and where they lead” or F8, who says that one should “learn from the map, facts are entered once again.” The student’s comments also indicated that the concept maps evoke concept images that contain conflicting pieces of information; consider the comment made by M5: “It only causes confusion... does not bring order, to me, it is just confusing.”

When the preservice teachers commented on their maps, they first mentioned the hierarchical maps, which was often an end product of their work method. The students rarely mentioned concepts and statements which were present on the map; instead, they tended to describe their experience of drawing the maps, which they often perceived to be a demanding task. F5
puts this view forward: “The concepts which are used are extremely abstract. It is difficult to see what can be included and what can’t”. M4 writes: “it is difficult to get started”. F6 comments: “Tiresome is my first thought. Unfamiliar is the next thought.” F1 states: “It is hard to know what to include and where to set the limit.” F4 comments: “At first it is difficult, otherwise it is fun.”

The students’ comments indicate that most of them thought that the hierarchical maps gave a clearer overview of the contents of the map (even if they often mixed general and specific concepts). This view is held by F2, for example: “It was good to subsequently arrange the concepts hierarchically … since I am thus able to see everything in an organized manner.” M6 agrees: “… it became clearer when it was redrawn. Additional relations were identified.” F16 writes: “The hierarchical format is clearer.” But there are also comments to the effect that it was more taxing to draw a hierarchical map, as indicated by F17’s comments: “It was difficult to draw hierarchical maps because I did not really know in which direction the arrows should be drawn.”

When the students drew the hierarchical maps they were often based on the freely constructed map; the contents were thus further processed, as indicated by F8: “It was good that we were able to redo them, firstly because it is easier (naturally) to understand the new redrawn version, but also because you remember and add several things which you did not in the previous one.” M8: “I tried to arrange 3 and 4 [refer to the two hierarchical maps] according to general concepts and then according to sub-concepts in decreasing order. At the same time I corrected (added, subtracted or re-formulated some things).” F5: “When I drew the maps in hierarchical format I got a clearer image and thus changed some sections.”

Just as in the previous group, there are clear indications that the students find it easier to interpret each other’s hierarchical maps. This is observed in the comments which F9 makes about the maps drawn by F3: “When I look at it [the freely formatted map for y=x+5] I become confused and can no longer follow her train of thought. The same applies to [the freely formatted map for y= \pi x^2], one is forced to ask her to explain it. It is easier to follow her train of thought in [the hierarchical map for y=x+5] and it appears to be more structured. It is also easier to interpret [the hierarchical map for y= \pi x^2]; one gets the impression that she knows what she is doing.” From the comments which F15 makes on F2’s maps: “… It is very easy to get a clear perspective of the hierarchical maps. The maps simplify the process of detecting the connections. The first and second maps [freely formatted] may appear to be rather confusing. The structure of maps three and four [hierarchical maps] is better.”

The students in the second group had a longer continuous period of time in which to draw their maps than the students in the first group; they were
thus able to draw more detailed maps. There were practical problems when the maps were to be constructed in hierarchical format. As F11 writes: “it is difficult to organize space and surface. One identifies several things which are related and draws a line over half of the paper”; or as F9 puts it: “New thoughts crop up now that we are to begin from scratch. It means that we make good use of the scrap paper and thus quickly abandon the attempt to draw it again on a new sheet of paper.” This is illustrated with the map which F11 submitted as her hierarchical map:

The map drawn by F11: It clearly indicates that the hierarchical structure was lost when F11 began to draw nodes in connection to \( \pi \), which gave rise to a significant portion of the map (the paper was not big enough to make further additions to the hierarchical structure in descending order).

The map illustrates a problem which prevents the student from drawing a map with a hierarchical structure. There was insufficient space for the map on the paper and so the student attempted to fit the map on the sheet instead of continuing on a new sheet of paper.

The map which F11 drew is unusual in that only one concept is directly linked to the statement \( y=\pi x^2 \). Such a phenomenon is rarely observed in the students’ maps (which typically have 3-4 links to the mathematical statement). F11 seems to believe that \( y=\pi x^2 \) is only represented by the function concept. In many cases, the map contains trivial information. There are also procedural undertones and parts of an algorithmic nature, for example “square of” gives “2 answers” etc. (as described in McGown & Tall, 1999; Williams, 1998). It appears that F11 makes free associations without defining a clear
structure of meaningful concepts and relations on the map. As in the case of the node “x^2”, which seems to have the highest number of links on the map and is surrounded by many trivial nodes and links, e.g. “xx”.

7.2.3 Some observations about the preservice teachers’ maps

The statements on the freely formatted maps are (as in the first group) placed at the center of the map; the map has a web-like structure, but very few cross-links (e.g. those drawn by F3, F5 and F18). The maps are less detailed than the hierarchical maps which often served as a draft of the information which would later be included in the hierarchical maps. The hierarchical maps tend to branch off into substructures which often have few cross-links (as in those drawn by F2, F8 and F16), implying that the concepts which are perceived to be represented by the statement often develop separately from each other. Although the maps have a hierarchical form, the conceptual structure is less hierarchical; thus more general and overlying concepts tend to be mixed with more specific concepts (a completely hierarchical conceptual structure is however not possible, since the two maps are derived from specific examples). Furthermore, it is clear that when the preservice teachers construct maps for y=x+5 and y=πx^2 at the same occasion, it influenced the maps contents and structure (as indicate by the maps and comments of F1, M8 and F16).

When one studies the students’ maps, it becomes evident that they do not always use or master mathematical terminology (Grevholm, 2000a, 2000b). For example, F1 links “y=πx^2” to “algebraic expression”\(^{19}\), F2 refers to x and y as “two unknown numbers” (in relation to y=πx^2), F4 links “y=x+5” to “algebraic expression” and F8 describes the exponent in πx^2 as “an elevated number”\(^{20}\). The preservice teachers also raise matters which are trivial, to say the least, when they comment on their maps (as also described by Williams, 1998). So M1 states that y=x+5 is “easy to draw”, F2 states that x^2 is x “multiplied by itself”, F8 writes the Greek letters “ε, α, β” in relation to π and F18 states that π may denote a plane in space. In addition, there are elements of an “algorithmic nature” (Grevholm, 2000a, 2000b; McGown & Tall, 1999; Williams, 1998) in which F3 states that y=πx^2 has two roots \(x = \pm \frac{y}{\sqrt{\pi}}\), F8 establishes a relation to the area of a rectangle “A=ab” and a triangle “a=bh/2” (in relation to y=πx^2) and F5 states that “the derivative” of y=πx^2 gives a “minimum point”.

A few students establish links between teaching and learning in their maps (similar to Doerr & Browers, 1999): F5, for example, states that

\(^{19}\) Giving the impression that she understand y=πx^2 to be an algebraic expression.

\(^{20}\) “Ett upphöjt tal” (swe).
“$y=\pi x^2$” is a “second degree function” which is “difficult” and that “primary school students cannot manage it”. This is in contrast to “$y=x+5$” which she labels a “simple function” which “is used in primary schools.” The preservice teachers also expressed similar opinions on degree of difficulty without establishing any connection to learning: F3 states that $y=x+5$ is a “simple function” and M6 states that “$y=x+5$” is “easier” than “$y=\pi x^2$” (on his map for $y=\pi x^2$).

The maps also illustrate that students have incorrect impressions which are related to the function concept. M2 links “inverse” till “equation”, F16 states that $x=\sqrt[\pi]{y}$ is the inverse\(^2\) of $y=\pi x^2$, M17 links “equation” to “has derivative” and F13 links “$y=\pi x^2$” to “exponential function”. It is also observed that the preservice teachers experience difficulties in distinguishing between the function concept and the equation concept (similar to Leikin et al., 2001; Vinner & Dreyfus, 1989; Williams, 1998, to name a few). This is revealed on the maps drawn by F1 and F14, for example, where they have double links between “equation” and “function”. M8 places “function” as a sub-concept\(^2\) to “equation” in his hierarchical maps (or as F3 notes in her comments on her maps, “I am not sure of the difference between equation and function. They are basically the same thing to me”).

If one studies how the function concept is expressed on the maps then it can be stated that all\(^3\) of the preservice teachers in the group have the function concept in their maps. There are clear indications that the preservice teachers have compared their maps for the two statements $y=x+5$ and $y=\pi x^2$, before adding more concepts to each map (e.g. F2, M5, F16). Concerning classes of functions, it was observed that about half of the preservice teachers did not tie the function concept to any special function class (e.g. F1, M6, F18), while nearly a third give “second degree function” and “function” for $y=\pi x^2$ and $y=x+5$ respectively (M5, F14, F16). Other preservice teachers state that $y=\pi x^2$ is a “second degree function” and that $y=x+5$ is a “first degree function” (M4 and M8) or “linear function” (M2, F51, F17). It was noted that preservice teachers also associated $y=\pi x^2$ with other function classes, of which F13 names “exponential function”. Otherwise, the preservice teachers do not mention any properties of functions, implying that they view the statements $y=x+5$ and $y=\pi x^2$ as belonging to different categories of functions.

In those cases where for example the derivative is discussed (only a few of the

\(^2\) Without any reference to domain or codomain of the function.

\(^3\) M8 links “$y=f(x)$” to “function”; the concept map implies that M8 interprets the notation “$y=f(x)$” as an equation and thus views “function” as a sub-concept to “equation”.

\(^4\) When the preservice teachers in the second group had a longer continuous period to work with the concept maps, there may have been some conversations going on between some of the preservice teachers that could have had an influence on the contents on their maps.
students mention derivatives in their maps, among them M1, M4 and F18), it is not viewed as a property (Slavit, 1997; Williams, 1998) which gives rise to a class of functions, but instead as a procedural skill (Hiebert & Carpenter, 1992) in the form of a calculation process. For example, just as M2 states that the derivative determines the slope, F5 states that the derivative determines the minimum point; or F18 states that derivatives are related to the slope of a curve. Derivatives are, however, more common in the maps which are derived from \( y=\pi x^2 \) (6 maps) than for maps which are derived from \( y=x+5 \) (4 maps). Furthermore, the preservice teachers often express the function concept as a "relation"\(^{24}\) (e.g. F1, F12, F16) or a "dependence"\(^{25}\) (e.g. F3, F10, F11) between the variables x and y, reflecting a process conception (Eisenberg, 1991; Sfard, 1991, 1992). None of the preservice teachers discusses domain or codomain in relation to the function concept.

The node which contains the function concept is usually connected to the statement from which the map is derived and often gives rise to an underlying structure with very few links to other sections of the map (som t.ex. F2, F9, F10) – even if there are students who integrate the function concept in their maps to a great extent (e.g. F1, M4, F14). Maps that contain fewer cross-links highlight to a lesser degree the manner in which the different concepts – which are represented by the statement – are related to each other. For example, F2 has the nodes "straight line" and "function" but does not state that the straight line is a graph of the function. Similarly, M7 links "curve" to "minimum value", but does not link "minimum value" and "function". Whereas preservice teachers who largely link the function concept to other nodes on the map, also link it to concepts which give the impression that they do not fully understand the function concept; F14, for example, has double links between "function" and "equation" (with link words "have a" and "determine" respectively), while F1 links "function" to "algebraic expression" (with link word "contains").

8 Evaluation – a subjective activity

The concept maps in the study are intended to be used as an open diagnostic tool in the qualitative study of people’s knowledge and understanding of concepts in relation to a mathematical statement. Studies in mathematics education which use different types of concept maps support the idea that concept maps possess a wealth of information on an individual’s conceptual understanding (Baralos, 2002; Doerr & Broers, 1999; Grevholm, 2000a, 2000b; Haseman & Mansfield, 1995; Heerer & Kommers, 1992; Laturno, 1994; Leikin et al., 2001; McGowen & Tall, 1999; Williams, 1998, 2003). Further-

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\(^{24}\) "Beroende" (swe.)

\(^{25}\) "Samband" (swe.)
more, they also attend to the manner in which a diverse conceptual interpretation is reflected in the maps (Baralos, 2002; Haseman & Mansfield, 1995; Laturno, 1994; McGowen & Tall, 1999; Williams, 1998, 2003).

Although the maps possess a wealth of information, it may be difficult to interpret them and understand an individual’s thought process during the construction of these maps. In this regard, Leikin et al. (2001) believe that it is necessary to supplement the maps with interviews in order to interpret the information they contain. I believe that it may be desirable to supplement the maps with additional information as to how they should be interpreted. However, the need to supplement the maps with more information is reduced when the maps are constructed in a more homogenous manner, i.e., with a hierarchical structure with labeled links.

Novak makes the following reflection with regards to the evaluation of the concept maps:

Although there is some subjectivity in scoring the maps, the great freedom given to individuals to demonstrate their idiosyncratic meanings for the subject matter removes an important source of bias and subjectivity that is present when the test writer chooses the specific content and form in which answers must be selected. (Novak, 1998, p. 194)

A degree of subjectivity is unavoidable when interpreting concept maps. I have mainly noted some of the principles of evaluation which were developed by Novak and his colleagues (Novak, 1998; Novak & Gowin, 1984) and based on Ausubel’s theory of assimilation. The validity of the evaluation principles is primarily studied in science education (Markham et al., 1994; Mintzes et al., 1999; Novak, 1998), but also in mathematics education (Laturno, 1994).

9 Discussion and conclusions

When the preservice teachers in the first group were instructed to draw a freely formulated map in relation to the mathematical statement \( y=x+5 \), they constructed their maps in many different ways and the structure of the maps varied. Nevertheless, there was a tendency to surround the statement with nodes much like the central hub of a wheel. While sections of the map often had a web-like structure of varying density, the structure varied from one in which the nodes lacked many connecting links to one in which they were linked to several nodes so as to resemble clusters, such as those described in Doerr and Browsers (1999). There was rarely any hint of a hierarchical structure in the maps; neither did any of the students draw a hierarchical map with reference to general, overlying or specific concepts.
A comparison of the freely formulated maps and the hierarchical maps revealed that the students in the first group raise different concepts and connect them differently on their two maps. This is not surprising, considering that none of the students had drawn their first map with a hierarchical structure, but were subsequently directed to draw the second map with a hierarchical format without access to their first maps. But although the number of links and nodes is fewer in the hierarchical maps (in the first group) and the number of links to the statement \( y=x+5 \) in particular, similar “themes” tend to be found on the two maps; i.e., if a connection to a line, equation or function exists on one map then it will usually exist on the other map, although in a somewhat modified form. Furthermore, even if the number of links are reduced in the hierarchical maps – compared to the freely formulated maps – there is no clear trend with respect to the number of cross-links on the maps, which increased or decreased from one individual to another. There were generally very few cross-links on the preservice teachers’ maps, which was in agreement with William (1998), and Doerr and Browsers (1999).

Concerning the hierarchically formulated maps which the students drew for the mathematical statements \( y=x+5 \) and \( y=\pi x^2 \) respectively, it was observed that the students often mixed general (i.e., more inclusive) and specific concepts or used them interchangeably. Some students even tended to try forming sentences with nodes and links without considering a conceptual hierarchy, which may be attributed to their inexperience in drawing hierarchical maps. Experience in using hierarchically constructed maps in science education shows that the maps were more successfully utilized as the student’s (the subject’s) experience in drawing that type of map grew (Novak, 1998; Novak & Gowin, 1984).

It was also discovered that the students found the task of drawing a hierarchical map to be more challenging than that of drawing a map according to a structure of their own choosing. One of the reasons for the students’ mixing of inclusive and specific concepts could be that the task of drawing a map based on a mathematical statement places greater demands on a conceptual understanding of the various concepts which the statement is perceived to represent and their relation to each other, rather than what a map which is only derived from a specifically stated concept does. The various concepts which the statement is thought to represent evoke concept images of different concepts (Tall & Vinner, 1981) which, when joined together, become more comprehensive and thus more difficult to process than would have been the case if the map had been derived from just one concept.

Opinions on the work methods adopted by the students when they drew their maps were more prominent in the written comments which were submitted by the first group. The students generally preferred the method which they used during the construction of the freely formulated map. Their com-
ments reveal that they often viewed the maps as tools to be used to illustrate ideas in a less structured format (e.g. a sort of mind map, Buzan, 1995) in which the maps were primarily of value to the individual. There were clear indications that the students themselves preferred their own unstructured maps; at the same time, they preferred their colleagues’ hierarchical maps since they perceived these to be easier to follow and understand. The work method which was used in the second group received more positive comments from many students. The contrast between the working methods which were applied to the construction of freely formulated and hierarchical maps were not raised. Instead, the entire project was viewed as a work process in which the goal was to draw a map with a hierarchical structure, which the students perceived to be clearer. This structure subsequently became more comprehensive than the freely formulated map which they had initially drawn. This could imply that the work method promoted progressive differentiation and integrated reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978). Many of the students’ written comments revealed clear perceptions that the maps promote metacognition and have a mediating role, but there are also signs that the maps evoke concept images with conflicting segments of information.

It can also be mentioned that the preparation of hierarchical maps by hand is fraught with practical problems. The more time the students were allotted to draw the maps, the more detailed the maps became and the greater the problems faced by the students (who often limited a map to one sheet of paper) when developing the maps with a prescribed hierarchical structure.

The maps which were drawn by the students reveal that they link the statement to a number of nodes. These nodes seemed to be the most strongly represented concepts in relation to the statement (Goldin, 2002; Goldin & Kaput, 1996; Hiebert & Carpenter, 1992), and thus mainly in the hierarchical maps. When the students drew maps with hierarchical structures the number of links from the mathematical statement was often largely reduced and concepts connected to the statement, which the statement was understood to represent, increased – in comparison with the freely formatted maps – and thus generally allotted more space on the map. Nonetheless, this was more evident in the first group of students who had not had access to their freely formatted maps. Moreover, one can observe that the nodes on the maps do not always represent the statement the map is derived from, but may instead represent a concept to which the preservice teachers establish a strong association in relation to the statement (e.g., $\pi$ in relation to $y=\pi x^2$).

There were clear indications that concepts, particularly those which represented the statement, were often developed independently of each other, i.e., they had few cross-links, indicating a less developed ability to relate different concepts to each other in a meaningful manner (similar tendencies were iden-
tified in, Doerr & Browsers, 1999; McGown & Tall, 1999; Williams, 1998). This may prevent the preservice teachers from building a rich conceptual structure with meaningful concepts (Ausubel, 1968, 2000; Ausubel et al., 1978) and relations which form the basis of learning with understanding (Hiebert & Carpenter, 1992).

The maps further indicate that many students either did not use mathematical terminology or used them incorrectly (Grevholm, 2000a, 2000b); this indicates that the preservice teachers’ language is less developed, which can constitute an obstacle to meaningful learning (Ausubel, 2000). Elements that express procedural knowledge and skills of an “algorithmic nature” occur frequently, in agreement with results presented in Williams (1998) and McGown and Tall (1999). The maps may also contain elements which are completely trivial at the expense of important concepts and relations between them, which indicates root learning with a lower degree of conceptual understanding (Ausubel, 1968, 2000; Ausubel et al., 1978; Hiebert & Carpenter, 1992).

The preservice teachers rarely relate to teaching and learning in their maps. This may be surprising, since their mathematics courses all contain parts related to mathematics education. Particularly as the characteristics of the two statements $y=x+5$ and $y=\pi x^2$ make them suitable for connection to different teaching scenarios which the preservice teachers will face as inservice teachers.

Barely half of the preservice teachers in the first group included the function concept in their maps, where the function concept is not usually a well-integrated concept. Instead, it often takes the shape of a node with a few links which are not connected to other sections of the map. Different properties of functions, where the students mention e.g. whether it is differentiable or has an inverse, are only affected in exceptional cases and even then, mainly in the form of procedural skills. No student state that $y=x+5$ is a linear function, while connections to straight lines were rather common.

All of the preservice teachers in the second group included the function concept in both of their maps, where some categorical groupings of function were also present in the maps, as opposed to the case of the students in the first group. There are clear indications that the maps which the preservice teachers in the second group drew for the expressions $y=x+5$ and $y=\pi x^2$ have affected the content and structure of their maps, when constructed at the same occasion. Concepts that were raised in the map for one statement occur quite frequently on the map that was drawn for the other statement. The result suggests that the statement $y=\pi x^2$ gives rise to mental images which contain the function concept to a greater degree than $y=x+5$ does. This view is supported by Tall and Bakar (1992), who show that first-year mathematics students largely perceive $y=x^2$ to represent a function. Several misconceptions can be
identified when the function concept is found on all maps; for example, inverse and derivative of a function are linked to “equation” and the preservice teachers are uncertain as to the relations between “function” and “equation”. It is also found that the function concept is not properly integrated with other concepts on the maps drawn by the preservice teachers, this is often a consequence of that they are not observing the functions different properties and their relations to other concepts. In those cases where the students give a concept, or some characteristic of a concept, they do not usually observe the relations to the function concept. Furthermore, the preservice teachers often express the function concept as a dependency between the variables x and y. In some cases they give somewhat more elaborate explanations and state that an x gives one y, but none of the students discuss domain or codomain in association with the function concept.

The perception of the function concept which is illustrated in the concept maps drawn by the preservice teachers contrasts highly with the idea of functions playing a central and unifying role in mathematics (Carlson, 1998; Cooney & Wilson, 1993; Selden & Selden, 1992). The concept maps seem to reveal a need for the students to reflect upon relations between the concept of function and other concepts in mathematics, in an attempt to facilitate more integrated knowledge structures and promote the preservice students understanding of the concept of function.

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Paper III
Preservice teachers’ views on three mathematical statements: A case study regarding the concept of the function

Örjan Hansson

Three preservice teachers participate in a case study during the final mathematics courses of the teaching program. The purpose of the study was to investigate how preservice teachers at various stages of competence in mathematics view the statements \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \), with particular emphasis on the manner in which the function concept is expressed. None of the preservice teachers in the case study describes the concept of function in a way that is consistent with the definition of function. As the performance of the preservice teachers in mathematics decreases, a process-based conception of the function concept is becoming more prominent. The preservice teachers’ knowledge structure of different concepts which the statements are perceived to represent also seem to become more compartmentalized; the number of meaningful relations to other mathematical concepts becomes less, as their level of performance in mathematics fall. Preservice teachers often use geometrical interpretations when they are describing various properties of a function in relation to the statements. The form of the statements influences the concepts that they are deemed to represent. Cognitive obstacles and prototypes seem to exist in relation to the function concept. The preservice teachers often seem to lack a mathematical language to describe the properties of functions, and use function classes and categorizations to a low degree when discussing the concept of function.

1 Background

Functions are present within all areas of mathematics. The function concept has a comprehensive network of relations to other concepts (Blomhøj, 1997; Eisenberg, 1991; Selden & Selden, 1992). It is imperative that preservice teachers have a good understanding of the function concept, partly in order to become successful in their studies in mathematics (Carlson, 1998; Eisenberg, 1991, 1992; Tall, 1992, 1996; Thompson, 1994), but also to be better equipped to manage the concept in the role as a teacher (Chinnappan & Thomas, 2001; Cooney & Wilson, 1993; Even, 1990, 1993; Even & Tirosh, 1995, 2002; Fennema & Loej, 1992; Thomas, 2003; Vollrath, 1994). In this regard, it is relevant to study preservice teachers’ views on mathematical statements that can be connected to different topics and levels of mathematics, at tertiary level as well as to their future supervision, in order to examine how the function concept is expressed.
Preservice teachers’ views on the statement \( y=x+5 \) and the way in which the function concept is expressed have previously been investigated by Hansson and Grevholm (2003). The result indicates that preservice teachers’ function concept was represented by a less developed cognitive structure than was desirable in a future teacher. Hansson (2004) states that the function concept, in relation to the statements \( y=x+5 \) and \( y=\pi x^2 \), tend to have few relations to other mathematical concepts that could prevent preservice students from building conceptual frameworks that are rich of meaningful relations; thereby creating poorer conditions for meaningful learning (Ausubel, 1968, 2000; Ausubel, Novak & Hanesian, 1978) and learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992).

2 Aims of the study

The mathematical statements \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \) may be perceived to represent a number of concepts which are related to different areas of mathematics the preservice students have come in contact with, both before and during their time in the teaching program. In particular, the three statements may be thought to represent real valued functions of a real variable. In this regard, it could provide the opportunity for the students to link them to different classes of functions and identify their various properties and relationships to other concepts.

The purpose of this study is to investigate how the function concept is expressed in relation to \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \) for preservice students at various stages of performance in mathematics, by means of a case study. What properties the students identify and how relations between the function concept and other concepts are presented will be examined.

3 Theoretical framework

In the chosen theoretical framework, knowledge is represented internally and described in terms of the way an individual’s mental representation is structured. Internal representations can be linked, metaphorically, forming dynamic networks of knowledge with different structure, especially in forms of webs and vertical hierarchies (Hierbert & Carpenter, 1992). Understanding grows as the networks, the cognitive structure, become larger and more organized where existing networks influence relationships that is constructed thereby helping to shape the new networks that are formed. Hiebert and Carpenter (1992) describe the construction of larger and more organized networks of knowledge as learning with understanding. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. A similar notion is described by Ausubel (2000) in the terms of meaningful learning, as opposite to
rote learning “only in rote learning does a simple arbitrary and nonsubstantive linkage occur with pre-existing cognitive structure.” (Ausubel, 2000, p. 3)

A central part of Ausubel’s assimilation theory of meaningful learning (Ausubel, 1968, 2000; Ausubel et al., 1978) is the idea that new meanings are acquired by the interaction of new, potentially meaningful ideas (knowledge) with what is previously learned. This interactional process results in a modification of both the potential meaning of new information and the meaning of the knowledge structure to which it becomes anchored. The process of assimilation results in progressive differentiation in the consequent refinement of meanings, and an enhanced potential for providing anchorage for further knowledge. When new knowledge is learned the already existing knowledge structures can recombine themselves and new and different meanings may develop and conflicting meanings may possibly be resolved through a process of integrative reconciliation.

The total cognitive structure that is associated with a concept in the mind of an individual is called a concept image (Tall & Vinner, 1981), including “all the mental pictures and associated properties and processes” (p. 152). A concept image is built up during a longer period of time through an individual’s experiences. When an individual meet an old concept in a new context, as in the case of function (Vinner, 1983, 1992), it is the concept image with all the implicit assumptions abstracted from earlier contexts that respond to the task. The portion of a concept image that is activated at a particular time is called the evoked concept image.

In an individual’s conceptual development Sfard (1991) suggests a process-object model, especially applicable to the concept of function (Sfard, 1989, 1992). The formation of an operational conception, as a process, precedes a later more mature phase in the formation of a “structural” conception regarding functions as objects. According to Sfard, both conceptions are essential and should coexist in a dual view of the function concept. In the transition from operational to structural conception Slavit (1997) suggests an emphasis on functions’ properties to enhance the development of a structural conception.

4 Method and procedure

The study is part of a larger study on preservice teachers’ understanding of the function concept. The preservice teachers in the study are in the sixth term of the teaching program and specialize in mathematics and science, grades 4 to 9. During the term, they are enrolled in the final mathematics courses of the program. Data collection takes place primarily at the end of the term, after a calculus course where the function concept has been a central concept.
The choice of the statements \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) enables the preservice teachers to relate to various concepts and areas in mathematics which they had previously encountered, but even others which they encountered during the teaching program. It is possible to let the statements represent concepts such as a straight line, parabola, hyperbola, equation, formula, proportionality, function\(^1\) and others. In connection to the function concept, the preservice teachers can identify various classes of functions related to \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \); these functions may be, e.g. linear, polynomial or rational, but may also be, e.g. continuous, differentiable, even or odd. The preservice teachers may also indicate whether the functions have inverse, asymptote, extremes or other properties which are treated in a calculus course. The statements also give the preservice teachers an opportunity to relate to future teaching situations on the function concept.

### 4.1 Method

In order to investigate the preservice teachers’ views on the statements, they were asked to answer a questionnaire both before and after the calculus course. The questions were open and read “We write \( y = x + 5 \). What does it mean?”, “What can you say about \( y = \pi x^2 \)? Please give as detailed an answer as possible” and “What can you say about \( xy = 2 \)? Please give as detailed an answer as possible.”. The questionnaire also contained a question which asked the preservice teachers to describe a function: “Describe, in your own words, your interpretation of the concept of function.” They further drew concept maps, for the statements \( y = x + 5 \) and \( y = \pi x^2 \) in order to describe their knowledge and understanding of the statements; the concepts they are deemed to represent, conceptions about these concepts and their relationships. Finally, the preservice teachers were interviewed to enable them to expand their answers.

The preservice teachers answered the questionnaire and drew the concept maps individually in order to facilitate a comparison of the differences in perception among them concerning the function concept and the mathematical statements. This was done in conjunction with their mathematics class, which enabled them to answer the questionnaires and draw the concept maps under similar conditions (this meant that an opportunity for the students to also draw concept maps for \( xy = 2 \) did not present itself, since it is a time consuming activity).

The interview was formulated such that the preservice teachers were each asked to comment on the answers that they had previously given on the two questionnaires. They were then allowed to supplement their answers and clarify their views on the function concept and the way it is expressed in rela-

\(^1\) If we assume that a domain and a codomain are given.
tion to the three statements. In order to gain a better understanding of their thought processes, they were also frequently asked follow-up questions based on their previous answers (Kvale, 1996). For practical reasons, the interviews were conducted during predetermined time intervals. This meant that it was sometimes necessary to limit the time for a topic to some extent and move on to the next question. The preservice teachers were previously acquainted with me, as I was one of their teachers. This contributed to a natural interaction between interviewer and interviewee.

Three preservice teachers were selected for participation in a case study. They were drawn from a large batch of data that was compiled on all preservice teachers in the sixth term of the mathematics and science program. The students were selected on the basis of their performance in the calculus course, based on the three-grade scale of the course: “high pass”, “pass” and “fail”. Consideration was also given to the performance of the three students in other mathematics courses during the term which clearly showed that the preservice teachers had distinctly different levels of competency in mathematics (it should be mentioned that preservice teachers with consistently high grades in mathematics constitute the smallest of the three categories). Consideration was also given to the concept maps that they had drawn, which were among the more extensive maps with respect to the number of nodes.

The study has been limited to the investigation of how preservice teachers view the function concept with respect to the three statements when they are enrolled in the last mathematics courses of the program. The nature of the statements is such that they may also be applied to different teaching situations to which the preservice teachers may be exposed to as inservice teachers. This study does not treat the way the preservice teachers handle the function concept in this case.

4.2 Procedure of the study

A questionnaire consisting of eight questions was distributed before and after the calculus course. (Questions 3, 4, 5 and 2c concerned the statements $y=x+5$, $y=\pi x^2$, $xy=2$, and an description of function, respectively. The first two questions on the survey\(^2\) concerned variables, equations, function and algebraic expressions, while the last three questions addressed the function concept.) The preservice teachers were able to answer the questionnaires.

\(^2\) The three introductory questions on the questionnaire are similar to the two introductory questions on the questionnaire which was used by Hansson and Grevholm (2003); otherwise, the contents of the two surveys are different. This has been done to create similar conditions for the groups of students which attended calculus courses when they answer the question on $y=x+5$, in case a comparison should become necessary. The survey, which was used in Hansson and Grevholm (2003) was originally formulated by Grevholm in conjunction with an algebra course for preservice students.
during lecture and spent 40 minutes on the task, with somewhat less time on the first questionnaire. A few days after the students had answered the second questionnaire, they drew concept maps for the statements $y = x + 5$ and $y = \pi x^2$ during a 90-minute lecture. The lecture began with an introduction to concept maps (a more detailed explanation of the process can be found in Hansson, 2004). Students who volunteered to be interviewed signed up for participation by writing their names on a timetable. Twenty students from the group were interviewed during a three-week period at the end of the term. The length of each interview varied, but they often lasted an hour. The interviews took place in a preparation room for mathematics and were recorded on tape. An interview included all questions on the survey. The recordings were usually supplemented with handwritten observations. Twenty four preservice teachers drew concept maps. Twenty two students had answered the questionnaire before taking the calculus course and twenty four answered it afterwards. The answers and the contents of the maps were compiled. All interviews were transcribed in their entirety. Three preservice teachers were subsequently selected for participation in this study as outlined above.

During the transcription process, sounds such as “hmm”, “ee” etc. have been removed according to recommendations presented by Kvale (1996, 1997). Furthermore, periods, commas, question marks etc., were inserted to make the transcription easier to understand. Selection of those sections which were raised in the interview took place after the transcription of the interview had been read repeatedly and sections of the tape listened to; recollections from the interviews and notes from the interview were also used. Sentence interpretation and focusing took place during the analysis of the interview according to the method prescribed by Kvale (1996, 1997).

4.3 The group of preservice teachers

The preservice teachers, which participated in the study, were enrolled in a four and a half-year teacher training program in mathematics and science, grades 4 to 9. The training program’s duration was four and a half years. The group consisted of 25 students of which women were in the predominant majority, and included all\(^3\) preservice teachers who were specializing in these subjects in the sixth term of the program. At the end of the term they would have completed the required 30 weeks of full-time studies in mathematics, of which a third would have been dedicated to mathematics education.

Before reaching the sixth term, the preservice teachers had studied mathematics in the first and third terms of the program. In the first term, they

\(^3\) The teacher preparation program had relatively few students that specialized in mathematics and science. The study was conducted at a tertiary institution with 9,000 students, half of which were enrolled full-time.
enrolled in the introductory course in mathematics which consisted of five weeks of full-time study. In the third term, the preservice teachers took a course in algebra and number theory, which together constituted 10 weeks of full-time studies. They then took the final mathematics courses in the sixth term of the program, which included statistics, calculus and geometry, corresponding to 3, 5 and 7 weeks of full-time work respectively. Applicants to the teaching program in mathematics and science, grades 4-9, should have successfully completed a program in mathematics and science at upper secondary school, or the equivalent.

5 Results

Three preservice teachers, Emma (F14), Nils (M6) and Vera (F8) participated in the case study. Their level of performance in mathematics during the sixth term can be arranged in descending order as follows: Emma, Nils, Vera. Their answers to the two questionnaires and the corresponding sections of the interviews have been compiled and are presented below. For each preservice teacher who participated in the case study there are two answers for questions 3 to 5 and an account of the interview in conjunction with the questions. The preservice teachers’ concept maps can be found in an appendix to the paper.

5.1 A case study with Emma

5.1.1 Answers to the questionnaires

3. We write y=x+5. What does it mean?

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any value can be assigned to x, and y will “automatically” be obtained. y is always greater than x by 5.</td>
<td>y is greater than x by 5. Here, one must personally decide x in order to know which y-value corresponds with this x-value.</td>
</tr>
</tbody>
</table>

4. What can you say about y=πx^2? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>y has the same value when one chooses a positive or negative x. It is the absolute value which matters.</td>
<td>y is greater than x, how much greater depends on x. The greater x is, the greater the difference between the two variables. π is a constant of proportionality. y is proportional to the square of x. If x is the radius of a circle then y</td>
</tr>
</tbody>
</table>

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4 Naturvetarprogrammet.
5 A more comprehensive analysis of the concept maps will be conducted in a planned further developed version of the paper.
5. What can you say about \( xy=2 \)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>One should choose 2 numbers which give a product of 2.</td>
<td>If one multiplies two independent variables then the product will always be 2. One must determine the value of one of the variables, e.g. ( x ). One then has to solve for ( y ) in order to obtain a value for ( y ). ( y = \frac{2}{x} ). You get pairs of numbers: an ( x )-value corresponds to a ( y )-value.</td>
</tr>
</tbody>
</table>

The answers are generally more detailed on the second questionnaire; they are also different in character. The answers to question 3 are more or less equivalent to each other, while the answers to question 4 differ and discuss completely different aspects (e.g. absolute value and area calculation). The answer to question five is much more detailed on the second questionnaire and includes much of the previous answer.

Emma discusses the variables \( x \) and \( y \) and their, mainly numerical, relations on the first questionnaire (with the exception of question 5) and to a higher extent on the second. It can be noted that Emma mentions the concept of proportionality in her answer to question 4 on the second questionnaire, but not in question 5. She also describes the difference between the variables \( x \) and \( y \) in questions 3 and 4, as opposed to question 5. Furthermore, pairs of numbers are mentioned in question 5 but do not lead to a geometric interpretation of the statement (geometric interpretations are generally not mentioned in any of the answers, other than in the form of a circle in question 4).

Question 5 on the second questionnaire begins with a contradiction: the first sentence contradicts the second and the third sentences. Question 4 also begins with an incorrect statement (since \( y \) is less than \( x \) when \( 0 \leq x \leq 1/\pi \)).

5.1.2 The interview coupled to the questionnaires

Emma read her answers to question 3 and realized that the answers were similar. She thought about whether they should be supplemented with anything “graphical”, before saying that \( y=x+5 \) was the equation for a straight line. She also mentioned that “pairs of numbers” arise because “if you choose \( x \) then you get \( y \)”, but still adds that \( y=x+5 \) is a function, referring to a previ-
ous question on the function concept. Emma did not know more to say, but stated that she had now said discussed the issues she had wanted to mention in connection to question 3.

Emma moved on to question 4 and read out the answers on the questionnaire (without correcting the statement that \( y \) is greater than \( x \)), adding that \( y=\pi x^2 \) is “of course also a function. A quadratic function…”; at this point, she mentioned a function class in relation to the function concept, in contrast to the previous statement. She subsequently tried, on her own initiative, to determine the appearance of the graph. Excerpts of the interview are included here:

E: Let me see here, what does \( \pi \) do? Yes, what does \( \pi \) do? … I just asked a question to which I don’t know the answer. [Laugh]
I: What happens when you have \( \pi \) there, in front \([\text{of } x^2]\)?
E: Well, then… in that case one assumes … one cannot begin from … well the parabola then? And … is it that which rises and thus falls along the \( y \)-axis?
I: You need to clarify. Rises and falls?

It is apparent that Emma does not realize how the factor \( \pi \) affects the shape of the parabola. In the following extract she begins with a parabola which is defined by \( y=x^2 \) in order to explain how it changes if it is multiplied by \( \pi \).

E: So if you have the parabola, i.e., when you begin with the parabola, then it will be at the origin. Then if you put a constant in front of it then it will be at another point on the \( y \)-axis.

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Emma went back to what was previously discussed about functions in the interview in connection to question 2c, where she should have described a function in her own words. On the first questionnaire, she answered “An expression with two variables (e.g. \( x \) and \( y \)), for each value of \( x \), there is just one value of \( y \)” and on the second questionnaire, “An expression which contains two different variables. You can draw the graph of the function in a coordinate system which shows the different solution pairs.”

In the interview, Emma said that the condition “for each value of \( x \), there is just one value of \( y \)” is “the theoretical stuff which one has learnt, has been drilled into us in school,” which clearly indicates root learning (Ausubel, 1968, 2000; Ausubel et al., 1978). Emma stated that “both [answers] felt good” and that “solution pairs” in the second questionnaire reflected “the same idea” as the answer on the first questionnaire, which according to Emma implied “that there are numbers then, which belong together in pairs”. (Later, in connection to question 3, she said “pairs of numbers” instead of “solution pairs”.) What Emma meant by “expression” in the questionnaire, was never clarified but many preservice teachers in the group incorrectly used the term “expression” when they actually meant “statement” in questions 3-5. When Emma subsequently raises the issue of “solution pairs” it indicates that she (with “expression” signifies an equation with two variables and) apparently makes a connection between equation and function. This is also evident in the concept map which she drew for \( y=x+5 \), which had double links between “function” and “equation” (with labels “have a” and “determine” respectively). Otherwise, Emma does not mention domain or codomain in her answers on the questionnaires; neither does she do so in the interview, in conjunction to question 2c (which is similar to the results from Tall & Bakar, 1992; Even, 1993).
Emma’s interpretation was that $\pi$ caused a vertical translation of the parabola. I informed her that this was not the case. She thus tried to explain the effect on the graph of $y=x^2$ with multiplication by $\pi$, considering whether the parabola’s “gap” would be changed – referring to the distance between points on the parabola which have the same $y$-value. She rejected her line of reason, saying that it is the exponent which determines “the gap”\(^7\). Emma used $x^4$ as an example: “I mean, if you have $x^4$ instead [of $x^2$], then it would look different. They do not have the same width” (thus making a correct observation with respect to the graphs of $x^2$ and $x^4$). Emma could not decide how the graph would be affected by the factor $\pi$ before she had plotted the coordinates and sketched the graph.

Before Emma left question 4, she commented on her answer on the second questionnaire by saying that her first statement (i.e., “$y$ is greater than $x$, how much greater depends on $x$”) relate to functions, and she explains this by stating that “Assign any value to $x$ and you get a corresponding value for $y$”. Furthermore, she said that by mentioning proportionality, on the second questionnaire, she also relates to functions (a viewpoint which is shared by Vollrath, 1986, and Leinhardt, Zaslavsky & Stein, 1990, as preliminary stage of the function concept) and points out that it can be visualized in a system of coordinates.

Emma then continued to read her answers to question 5 (without noting the contradictions to the answer on the other questionnaire). Concerning the answers on the questionnaire she spontaneously raised the relation between function and equation, as follows:

E: (...) You get pairs of numbers: an $x$-value corresponds to a $y$-value [completed the last sentence in the answer on the second questionnaire]. So it is also a function. Although it is written in a different form than I am used to. But it is... well... yes, exactly, it is an equation. But one says... yes, what does one say? A function’s... no, an equation of a graph... they... they belong together. Was this what we had problems with, when we were supposed to draw those thought maps\(^8\) the other day …

I: I see.

E: ... to relate it to that business of equation and function... That it is [pause]... If you have an expression, then you can of course call it an equation. But you can also say that it is a function and that it ties the two together. Because we say the equation of the graph, or.. people talk about the equation of a straight line and ... so I didn’t quite express this on the thought maps. [Laugh]

\(^7\) Emma also mentioned “the gap” on her concept map and linked it to $\pi$. Nevertheless, she rejects the connection in the interview.

\(^8\) Emma referred here to the concept maps which the preservice teachers drew for $y=x+5$ and $y=\pi x^2$ in the days leading up to the interview. Emma’s map for $y=x+5$ had double links between function and equation, which shows that she interprets the two concepts as being closely related.
Emma completed her reply by stating that $xy = 2$ is a function. In that regard, she felt that she was more familiar with the explicit form $y=2/x$, which is also reflected in the answers to the second questionnaire. The implicit form $xy = 2$ is a “form” which Emma claimed to be “unfamiliar with”. It appears that $xy = 2$ and $y=2/x$ (as opposed to the two previous statements) evoked concept images containing conflicting information for equation and function, respectively (Tall & Vinner, 1981; Vinner, 1983, 1992). It was clear that Emma could not explain how the function and equation concepts were related to each other, stating, as she did, that “they correspond to each other”, (Even, 1993; Grevholm, 2002; Leikin, Chazan, Yerushalmy, 2001; Tall, 1996; Thomson, 1994; Vinner & Dreyfus, 1989); this phrase was also used when she single-handedly made up incorrect expressions such as “the equation of the graph”$^9$. (This illustrates a need for integrative reconciliation by related cognitive structures (Ausubel, 1968, 2000; Ausubel et al., 1978).) Emma seemed to make generalizations based on the equation of a straight line, since she states that “people talk about the equation of a straight line”, clearly indicating that she believed the solution set for an equation generally produces a graph of a function. The straight line equation appears to be a cognitive obstacle when Emma tries to describe relations between the concepts of function and equation.

I mentioned what Emma had discussed about equation and function and asked her to explain the difference:

E: Well, earlier I thought there was a huge difference, but I don’t think that anymore. It is more likely that it can be demonstrated; if there is an equation with two unknowns, then it can be shown in some way with the aid of a graph. That way you can extract - after drawing it - you can extract pairs of numbers. These contain all solutions, so to speak… If you choose an x-value then you get a y-value.

It appears that Emma likens an equation with two variables to a function. She makes a geometric interpretation of the solution set (which she calls a “graph”) and seems to believe that a pair of numbers (x, y) in the set of solutions to the equation, has the property that the x-value results in just one y-value, without realizing that the solution set may contain several different pairs of numbers with the same first component. Emma has apparently come to believe that the two concepts are approaching each other, and should thus refer to the period during which she took the calculus course and the period subsequent to that, since she had previously stated that she saw “an enormous difference” between the concepts.

When she was once again asked to attempt to further explain her view on function and equation then it became apparent that her understanding of

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$^9$ Such an expression is not correct according to Swedish terminology.
equations had been influenced by equations with one variable (although she did mention the straight line equation – an equation with two variables – in her interview, which is something that she ought to have been familiar with for a long period). She said that she had now “changed her opinion” of what an equation was and that “it can also have two unknowns”. She then combined equations with two variables with the function concept, saying that “they [two concepts] begin to approach each other”. Her very language indicated that Emma could not distinguish between the two terms “function” and “equation”, since she sometimes said that there are “roots of the equation where the function intersects the x-axis” (similar results are presented by Even, 1993). The fact that it is possible to illustrate solution sets for equations with two variables in a system of coordinates seems to be something Emma relies on, when she combined the terms “equation” and “function”. She said that she did not have an answer that she was personally very happy with, but added after a short period of contemplation the following:

E: But a function is such that there is just one y-value for each x-value. Isn’t this what defines a function? On the other hand, there may be many x-values connected to a y-value… I have not really thought about what that may look like if the equations were drawn in a system of coordinates… properly. It may not be valid, that theory of there being just one y-value for each x-value. Although… I don’t know… It may be valid, as far as I can see at the moment. I have never seen anything else, but that does not mean that it doesn’t exist. I have never come across anything like that before.

Emma now considered the condition “there is just one y-value for each x-value” and began to see a difference between the two concepts (although other components of the function concept, such as domain and codomain are not mentioned, as in the results presented by Tall & Bakar, 1992). The interview revealed that she had not worked, or had no memory of having worked, with questions and problems which clarify relationships between the two concepts.

In order to help Emma give further explanations on the function concept I asked her “If one were to write y²=x, what could you say about that?” Whereupon Emma replied that one (positive) x-value then gives two y-values. She then looked at her sketch of the graph of y=πx² and notes that y²=x also results in a parabola, before saying, “although it is unsymmetrical10 around the x-axis”. I then asked “Is it a function, is y a function of x?” upon which she replied:

E: No, it is not, because the definition of a function. It is perhaps just that there

10 Emma understood that y²=x describes a parabola which is symmetrical about the x-axis (and unsymmetrical about the y-axis), since she sketched a correct parabola for y²=x during the interview. I assume that it was a slip of the tongue when Emma said “unsymmetrical about the x-axis”.

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is a... Now I put it together. One, there is only one... y-value to each x-value... and this is not the case in $y^2=x$. There are two y-values for each value of x.

It appears as if that Emma now try to use the definition of function in the first place and her concept image of function (Vinner, 1983, 1992) in the second place.

Emma seems to obtain greater clarity with respect to the relations between “equation” and “function”, indicating that integrative reconciliation (Ausubel, 2000; Ausubel et al., 1978) was taking place. The statement “Now I put it together” may be also be a sign of that. Emma also illustrated that a positive y-value gives two x-values in a correct sketch of $y^2=x$.

I then asked Emma to decide if x was a function of y in one of the questions 3 to 5, whereupon she immediately answered that this was the case for $y=x+5$ and $xy=2$. This, she said, was because $y=x+5$ gives a straight line and then a corresponding value of x for each y. She explained $xy=2$ by saying that one could solve for x and get $x=2/y$. (Emma based her answer on a geometric argument $y=x+5$, but not for $xy=2$, which is based on an explicit form and a symbolic argument. It later became evident that she did not know what the graph of $xy=2$ looked like before she had taken the time to draw it). Emma stated that she was able to solve for the variable x in both $y=x+5$ and $xy=2$ and in so doing, obtain one value of x for each value of y, but that this was not the case for $y=\pi x^2$, and then that x is no function of y. She subsequently used the term “inverse function”, stating that $y=x+5$ and $xy=2$ have inverse functions, but $y=\pi x^2$ does not. She also stated that $y=\pi x^2$ may have an inverse function (and thus introduces a restriction of the function) for an interval not including zero; she mentioned the intervals $x>0$ and $x<0$, among others.

I asked Emma whether she recognized the notion of “domain” and “range”; she recognized them and explained what they were. She said that she had not thought about it in relation with the statements in the three questions (corresponding with results from Tall & Bakar, 1992). Emma explained “domain” and “range” for $y=2/x$, but did not completely understand what the graph would look like and thus plotted some coordinates and did a sketch of the graph. She subsequently stated that the x and y axes are asymptotes, observing: “There was much which could have been said about it [y=2/x], that I had not said”. But in order to further discuss the statement she said that she wanted to get “some kind of lead” which she could build on. Emma did not consider that the statement represented concepts such as reverse proportionality and hyperbola, although she had mentioned proportionality and parabola in connection to the statement $y=\pi x^2$. It can also be noted that Emma did not discuss any function class, e.g. rational functions, or other properties in connection to the function concept, such as odd functions, or state that the function is the inverse of itself etc.
It was revealed during the interview that Emma believed that she was now “better at illustrating problems using images and not just words”. She also declared that she was nowadays more ready to “explain it with a picture”. She thought it was a process which probably began during the calculus course and now was continuing. An example of this, she said, was that she discussed “absolute value” in her answer to question 4 on the first questionnaire, while proportionality was discussed in the other. Proportionality was something she claimed to associate with the function concept because it can be illustrated in a system of coordinates.

5.2 A case study with Nils

5.2.1 Answers to the questionnaires

3. We write y=x+5. What does it mean?

<table>
<thead>
<tr>
<th>First questionnaire:</th>
<th>Second questionnaire:</th>
</tr>
</thead>
<tbody>
<tr>
<td>If we have a number x, then y is larger by 5, i.e. x+5.</td>
<td>That y changes depending on the value chosen for x. It is also a line</td>
</tr>
</tbody>
</table>

4. What can you say about \( y=\pi x^2 \)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire:</th>
<th>Second questionnaire:</th>
</tr>
</thead>
<tbody>
<tr>
<td>You insert a value for x, do the calculation and get a new value for y.</td>
<td>( y ) is dependent on x. ( y \geq 0 ) if ( x \in \mathbb{R} ). A curve which roughly looks like this</td>
</tr>
</tbody>
</table>

5. What can you say about \( xy=2 \)? Please give as detailed an answer as possible

<table>
<thead>
<tr>
<th>First questionnaire:</th>
<th>Second questionnaire:</th>
</tr>
</thead>
<tbody>
<tr>
<td>That the product of two numbers will be 2. Any number, so long as the product is 2.</td>
<td>( y ) is dependent of x or vice versa. ( y=\frac{2}{x} ). ( y ) can be any number apart from 0.</td>
</tr>
</tbody>
</table>

The answers to the two questionnaires have different characteristics. In the first survey, Nils considers the two variables x and y (except in question 5) and primarily discussed numerical aspects, before shifting to a more geometric interpretation in the second (again, with the exception of question 5), where he stressed the dependence between the variables in each statement.

It is worth noting that Nils brought up “dependence” between variables in question 4 of the second survey, drew a figure, specified the intervals of the
variables and used the term “curve”. This is an outlook with a strong association to functions, without explicitly naming it. He also makes explicit the domain in saying $x \in \mathbb{R}$, and range in $y \geq 0$.

5.2.2 The interview coupled to the questionnaires

Nils read out his answers to question 3 and stated that both answers are correct. He continued by saying:

N: One could even … well, carry over the x.
I: To give $y-x=5$.
N: Which results in a diophantine equation.
I: I see.
N: So there will be more of an equation there, but in its current state then it is more of a function. And it is a line, a straight line, in my opinion.

It is clear that Nils distinguishes between $y-x=5$ and $y=x+5$ in that he viewed the first as an equation and the second as a function\(^\text{11}\) in which the function concept is connected to an explicit form. It can also be noted that Nils viewed $y=x+5$ as a straight line and not as the equation of a straight line, which he nevertheless reflected on his concept map for $y=x+5$. It was also seen that Nils used different concept images (Tall & Vinner, 1981; Vinner, 1983, 1992) to interpret equivalent statements; he said that he “saw” different concepts, as illustrated in the following excerpt from his interview:

N: Yes, well, if you see it as a function then you can say that if you make x equal to 0 then it will intersect the y-axis at 5. Thus y would be equal to 5… and the slope of the line would be 1.

In conjunction with his understanding that $y=x+5$ represented a function, he applied a geometric interpretation and brought up intersection by referring to “the y-axis at 5” in a system of coordinates when x is equal to 0. It was noted that Nils in this context also viewed $y=x+5$ as a straight line; this became apparent when he specifies the slope of the line.

Nils continued to state values that the x and y variables could assume. He mentioned that y becomes 5 when x is 0, stating that y becomes negative when $x < -5$. He said that “one can set x to absolutely any value”, including “infinity” (which gives “infinity”, and would then represent y). Nils noted that the function was not bounded. In relation to this, he applied another geometric

\(^{11}\) The function concept was previously discussed in the interview in connection to question 2c, where Nils was asked to describe a function in his own words. He had answered the question on the first questionnaire by giving two examples: “E.g. $y=3x+1$ or $f(x)=4x^2+3x+1$”, while on the second questionnaire he wrote “A variable is dependent on another variable”. During the interview, he commented on the answers on his first questionnaire by saying that he had given “a good example” of a function and that they illustrate that “a variable which is dependent on another variable”, thus making a link to the answers in the first survey.
interpretation, stating that y=x+5 “rises completely and sinks below…” “in the form of a straight line. He tried to describe the properties of the function by combining both numerical and geometric interpretations. Although Nils claimed to view y=x+5 as a function, he did not use any terminology which was related to the function concept, such as domain, graph, increase and decrease – to name a few terms which are related to his comments. Neither did he highlight the fact that y=x+5 is a linear function or appear to master the terminology associated with the function concept when he attempted to describe the different properties of the function.

He proceeded to question 4 and read out what he had written on the questionnaires:

N: You enter a value of x, run the calculation and get a new value of y [he read the answer from the first questionnaire aloud]. Yes, but it is just as if they are dependent, like a function. And it is an x^2-curve, a variant of an x^2-curve.
I: Like the one you drew on the second questionnaire?
N: Yes, that can be checked if we insert … let us say πx^2. Set x = 1 so that y becomes equal to π … then we need to raise…
I: But what happens if you set x to 0?
N: Then y becomes 0… Yes, x and y, well… We say that y is 0, then I should move down, by the way [Nils realized that the graph was incorrectly drawn on the second questionnaire]. Yes, it should go through the origin, in that case.

Nils supplemented his answers to the questionnaires by stating that y=πx^2 is a function, basing it on the fact that the variables are “dependent” on each other. He also applied a geometric interpretation, perceiving y=πx^2 as “a variant of an x^2-curve” and using y=x^2 as the basis of his description of y=πx^2. It was revealed in the interview and by the figure drawn in the answer on the second questionnaire that Nils (like Emma) believed that the factor π means that the graph intersects the y-axis above the origin. It was not until he was asked to consider an x-value of zero that he realized that the origin is a part of the graph. Nils was then asked to draw the graph, upon which he stated:

N: (…) yes, it will be much more narrow [Nils drew the graph]… than a normal x^2-curve. So it is the actual width, if one could say so, which is changed when the constant is changed...

It is clear that Nils looked at x^2 in the right hand side and applied a geometric interpretation in the form of an “x^2-curve”, viewing y=x^2 more as a prototype in that context. This is also illustrated in the concept map which Nils drew for y=πx^2, in which he also stated that y=πx^2 is “narrower” than an “x^2-curve”, referring to “y=x^2”. (The concept map indicates that Nils appeared to realize how the shape of the graph of y=x^2 is changed if it is multiplied by π while he was drawing it. However, it is not apparent if he also believed that π results in a translation of the graph).
Nils outlined the values the variables could have, saying “one can set x to absolutely any value” and y is always positive “since it is the square of x”. He came back to his sketch of y=πx² to comment on the effect of x² on the graph: “it turns at the zero point and then goes upwards at both ends, if one could say so… so y can never be negative” and gave a description which, like the previous statement, was based on a geometrical interpretation and the values which could possibly be assigned to the variables, without referring to either domain or range.

Nils continued to comment on y=πx²:

N: Well, even that [meaning y=πx²] may be… you may write it as an equation. Although it is extremely tricky, that is of course my first impression. But I don’t know...
I: What...
N: Well, you can write it as an equation also. But I guess it is better as a function.

In Nils’ case “write it as an equation” meant gathering the variables on one of the sides of the equal sign (as was the case of y=x+5, in writing it as y-x=5). He explained that he was now of the impression that y=πx² represented a function:

I: You see it as a function?
N: Yes, automatically, when it is presented as y equals something with x. It is easier.
I: Did you also do this with the third question [y=x+5]?
N: Yes, as a function. One does this automatically. But when you think about it, what you can change… play around with, etc… But this [meaning y=x+5, in question 3] is significantly easier than that [meaning y=πx², in question 4].
I: The third one is easier?
N: Yes.
I: In what way?
N: Well, because it [y=x+5] is a straight line, so … it is easier to consider it than a quadratic curve [y=πx²]… But one would automatically think that it will be more difficult when one sees π as well. I mean, it really doesn’t matter whether there is a π or a 5. But once one sees π the automatic thought is that it is much more difficult.

Nils said he viewed y=x+5 and y=πx² as functions; he seemed to base this on the explicit form in which they are written when he said “yes, automatically, when it is presented as y equals something with x”. Nils also evaluated the degree of difficulty of the statements in questions 3 and 4, stating that y=x+5 “is significantly easier” than y=πx², based on the appearance of the graph, saying that a “straight line” is “easier to consider” than a “quadratic curve”. He thus appeared to base the difficulty of the statements on a geometric inter-
pretation. It can also be noted that Nils also used a different terminology, saying a “quadratic curve” instead of an “$x^2$-curve”. (He did not explicitly state that he meant a “graph of a function” when he used the term “curve”, but the concept map he drew for $y=\pi x^2$ nevertheless indicated that this was his intention, since it contains a horizontal sequence of nodes and links from the term “functions” to “$x^2$-curve”, via “table” and “graph”). Furthermore, it can be established that Nils found the presence of $\pi$ in the statement to be “much more difficult”, without realizing that $y=\pi x^2$ also gives the area of a circle with radius $x$.

I subsequently investigated Nils’ conception of the function concept – which he seemed to tie to an explicit form and a dependence between the variables – and in relation to $y=\pi x^2$ thus asked “…which of the variables is a function of the other, or what is your view?”. Nils replied that “$y$ is a function of $x$”. To get him to expand his view of functions I asked “can it be said that $x$ is a function of $y$?”. He then stated that “…there will be two values of $x$ for each value of $y$” and was uncertain of whether it was a function; nevertheless, he pointed to a dependence between the variables and did not rule out the possibility that $x$ could be a function of $y$:

N: When you set a value for $y$, you get a corresponding value for $x$, or two values for $x$. And then you may even set a value for $x$ and get a corresponding value for $y$… thus they could… they are thus dependent on each other …

Nils further contemplated the function concept; he stated that the statement $y=x+5$ in question 3 can be written in the form $x=y-5$, which in his opinion, meant that “$x$ is dependent on $y$” and thus that $x$ is a function of $y$. In the resulting discussion, Nils stated that $x$ ought to be a function of $y$ when $y=\pi x^2$, since one is able to write $x$ “freely”, i.e. solved for $x$, but said he was uncertain as to whether it is a function since it produces two values of $y$. Furthermore, with respect to the function concept, he stated that “a function is automatically perceived to be expressed as $y$ followed by the equal sign and then an expression containing $x$” and find it strange letting $x$ become a function of $y$. I therefore suggested that he study $x=\pi y^2$ and determine whether $y$ is a function of $x$. Nils solved for $y$ and obtained $y = \pm \sqrt{\frac{x}{\pi}}$ but cannot decide if $y$ is a function of $x$:

N: I don’t know if one can do that, so I can’t really… If $y$ is dependent then, because we get two answers or whether that is that. Because if this is the case, i.e., if this is allowed, then it is dependent on $x$.

Nils could apparently not remember the definition of function. I asked “What would such a function graph look like?” on him and his plots some values and draws a correct parabola for $x=\pi y^2$. But this did not enable him to decide whether $y$ is a function of $x$. Drawing the parabola apparently did not evoke
any mental images of function graphs which gave any insight into the fact that an x value corresponds to only one value on y (as opposed to Emma’s case).

The interview continued with Nils reading his answers to question 5. He demonstrated that x and y depend on each other by letting the variables assume values whose product was 2. He also said that one could “rewrite it [xy=2] as a function y=2/x”, stating that neither x nor y could be zero. In contrast to the previous statements, Nils did not apply a geometric interpretation; he was asked to describe the function graph. In reply, he plotted the coordinates and made a sketch of the graph, pointing out that the x and y axes form asymptotes.

I once again asked Nils to attempt to explain whether x is a function of y in questions 3 to 5. Nils replied that “it can be reversed”, explaining that one could solve for the variable x in y=x+5 and xy=2, continuing with:

N: (...) actually, it also should be, in question 4 [y=πx^2]. Although it is a square … It’s just that you get two answers. So yes. It should be, since you can make both [variables] independent, but then. I don’t really know if it is allowed, in the case where there are two answers. It is easier to see in that case [referring to xy=2], since it is just x and y.

It appeared that Nils’ view of functions was mainly based on a dependence between variables which is expressed in an explicit form. He did not take note of the definition of function, which was apparently unknown; in addition, questions on whether x is a function of y did not seem to lead to recognition of the concept of inverse function (in contrast to Emma).

5.3 A case study with Vera

5.3.1 Answers to the questionnaires

3. We write y=x+5. What does it mean?

<table>
<thead>
<tr>
<th>First questionnaire:</th>
<th>Second questionnaire:</th>
</tr>
</thead>
<tbody>
<tr>
<td>That something added with 5 becomes the sum y.</td>
<td>The number “hidden” behind x added with five gives the sum y.</td>
</tr>
<tr>
<td></td>
<td>It could also be said that y depends on x.</td>
</tr>
</tbody>
</table>

4. What can you say about y=πx^2? Please give as detailed an answer as possible.

<table>
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<tbody>
<tr>
<td>That a number raised to the power of 2 and multiplied by π will become y.</td>
<td>y depends on π and x^2.</td>
</tr>
<tr>
<td></td>
<td>y=πx^2 explains the area of a circle.</td>
</tr>
<tr>
<td></td>
<td>a=πr^2.</td>
</tr>
</tbody>
</table>
5. What can you say about \( xy=2 \)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire:</th>
<th>Second questionnaire:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two unknown numbers will become the sum 2.</td>
<td>If you multiply the variables ( x ) and ( y ) the sum becomes 2.</td>
</tr>
</tbody>
</table>

On the first questionnaire, Vera interprets the statements mainly as numerical calculations. She divides each statement into two expressions and begins each answer by calculating an expression, before stating that the result “becomes” the other expression. On the second questionnaire, she to a higher degree takes note of the variables in the statements, and a dependency relation in using the term “depends” (with the exception of question 5). Vera also includes new variables in conjunction with her interpretation of the statement in question 4 as the calculation of an area.

5.3.2 The interview coupled to the questionnaires

Vera read her answers to question 3 on the questionnaire. She stated that the answer to the first questionnaire and the first sentence on the second questionnaire are “exactly the same”; she proceeded to explain that \( y \) is dependent on \( x \) by determining \( y \) for different integers of \( x \). In relation to her explanation I ask “Does \( x \) depend on \( y \)?”, making her consider whether it could possibly be that way:

V: … Yes, it does. Yes, because we had set ... Well it must of course do that? Because if we had brought \( y \) over... and \( x \) had become free [explicit]… If we had done that then it would have been the same thing. It would have depended on… Yes, it does. Or, am I wrong?

Vera obviously tied the variables’ “dependence” to an explicit form, but was uncertain whether that was the case, i.e., whether \( x \) was dependent on \( y \). She was asked if there was anything else that she could say about \( y=x+5 \), to which she replied:

V: That... do you mean like a formula or something... or not? [Pause.] Oh! That it is a straight line!

Vera also associated \( y=x+5 \) with a straight line and then had nothing more to add.

The interview with Vera differs from those with Emma and Nils in that she was often unable to expand on her answers as much as the other two preservice teachers. Thus she was asked to examine her previous answers to the questionnaires, which include the function concept in order to allow her to give an additional response in relation to the statement \( y=x+5 \). The questions
concerned variable, equation, function and algebraic expression. The interview continued:

V: Yes, I believe I see it as an equation … I don’t think it is a function\textsuperscript{12}.  
I: No…  
V: [Laugh] … But it is, actually, if you think back to what it was. When I look at it I don’t think: “Oh! A function!”  
I: But…  
V: I can’t say why I don’t, but I don’t think about it. I would much rather see it as an equation.

Vera now chose to interpret the statement as an equation, which is something she did not mention in her questionnaire. She continued:

V: Wait … One could say that \(y\) is dependent on \(x\). So it should be a function as well. So let us say both function and equation.

It is clear that Vera by “\(y\) is dependent on \(x\)” means that \(y=x+5\) is a function. (She also expressed function on her concept map for \(y=x+5\), but then stated that \(y\) was dependent on \(x+5\), which indicates the dependence of variables to a lower extent). After this, Vera did not know any more to add regarding to the statement and was able to move on to question 4.

Vera read her answers to question 4. She commented on the answer on the first questionnaire by saying that the answer only states how to read \(y=\pi x^2\), and continued:

V: They are equal in weight, \(y\) is the one on that side [refers to the right hand side of \(y=\pi x^2\)].  
I: Hmm… on the right hand side, you mean…  
V: Yes, so \(y\) is the answer. So \(y\) is exactly the same as the result of the right hand side… And it says here that \(y\) is dependent on \(\pi\) and \(x\). It does, because if I change \(x\) then I change \(y\). And then I did not realize it then [on the first questionnaire], that it was the area of a circle. But it is.

Vera’s comments indicate that she viewed the statement as an equation, since she used a weight scale as a metaphor to state that the two sides “are equal in weight”. She also used a viewpoint which could be interpreted as being analogous to numerical calculations when she said that “\(y\) is the one on that side” (referring to \(\pi x^2\)) as well as “\(y\) is the answer”, thus expressing procedural knowledge (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). There was also an element of numerical interpretation when she said “\(y\) is

\textsuperscript{12} In question 2c of the first questionnaire, Vera was asked to describe a function in her own words and wrote: “curves which can be calculated.” On the second questionnaire, she wrote “something which depends on each other”. During the interview, Vera did not understand what she meant by her first answer, but settled on her answer from the second questionnaire. To further explain this answer she gave \(y=\pi x^2\) in question 4 as an example, saying that “\(y\) is dependent on \(\pi x^2\)”. She obviously did not perceive this to be a dependency which was first based on the \(x\) and \(y\) variables.
dependent on \( \pi \) and \( x \)” and that the \( y \)-value was dependent on the constant \( \pi \). Nevertheless, Vera notes the significance of the variables by stating “if I change \( x \) then I change \( y \”).

Vera did not apply any geometric interpretation to \( y=\pi x^2 \) in the form of a graph (in contrast to Emma and Nils). On the other hand, like Emma, she interpreted the statement as the calculation of an area and attempted to make it clearer by writing \( a=\pi r^2 \) and stating during the interview that \( a \) is dependent on \( \pi \) and \( r^2 \); in this sense, a viewpoint analogous with a calculation is once again applied, emphasizing an operation, rather than a relation between the variables \( a \) and \( r \).

Vera did not have any further comments on her answers to the questionnaires and thus again studied the previous questions on the first page of the questionnaire (which concern variable, equation, function and algebraic expression) before saying:

V: It is a function. And an equation as well, I would like to say. It has several… It consists of two variables, \( y \) and \( x \), as well as the constant \( \pi \).

It was not until Vera had read the previous questions that she stated that the statement represented an equation and a function. She had previously used expressions such as “weigh” and “dependent”\(^{13} \), which could be seen as metaphors which indicate a simplified view of the two concepts. The metaphors appeared to be at the center of her concept images for equation and function respectively (Tall & Vinner, 1981; Vinner, 1983, 1992), something which is thought to represent less developed knowledge structures. (A less developed knowledge structure seems also to be reflected in the concept maps which Vera drew. The concept maps do not have any cross links to connect different sections of the map; this gives the impression that the concepts have few meaningful relations and are thus represented by segmented knowledge structures that imply a less developed understanding, Ausubel, 2000; Hiebert & Carpenter, 1992; Novak, 1998.)

Vera read out her answers to question 5. She had written “sum” on the two questionnaires instead of product. I attempted to focus her attention on the issue:

I: The sum of two?
V: Hmm ... But actually [laugh]. If one could consider what we have previously discussed ... Yes, the sum of two ... here it is. Yes, because they should actually be equal in weight, where we can e.g. have \( x \) is 2 \( y \) gives 1, in order for them to be equal.

\(^{13} \) A dependency which is analogous to a numerical calculation is often observed, in which the result is dependent on the formulation of a mathematical expression, rather than on a variable which is dependent on another variable. This gives rise to a pronounced operational conception (Sfard, 1991, 1992) of function.
I: ... x. What did you say ...
V: So we can actually just set x to 2 and y to 1, which results in an answer of 2.
I: Yes.
V: And that ... I don’t know if one can say that x and y are dependent on each other here? Perhaps not?
I: Why not?
V: But it must be possible to say that, if that is the basis on which I... Because if x is free then we have 2/y... and when we change y then x is also changed.

It was observed that Vera did not realize that she had written “sum” instead of “product”. When she read her answers to question 5 she seems to take notice of the equality sign in the statement; she again used a “weight scale” metaphor, in which the left hand and right hand sides should “weigh the same”, giving the impression that the statement represents an equation. She suggested values for x and y which fulfilled the statement, but was uncertain as to whether it meant that “x and y were dependent on each other”. In order to confirm this she extracted x (to give x=2/y) and established that x changed for different values of y.

During the interview, it was revealed that Vera was uncertain as to how to comment on the statement in question 5 (which has the shortest answer of all the others on the second questionnaire) and considered whether she was on the “wrong track”. She once again explored the previous questions, before saying that x and y are variables, while xy=2 is an equation since it has “several unknowns”. She ended with the following:

V: And then I began to figure out... I remember that it is a function, since they are dependent on each other.

At this point, I asked Vera what she meant by writing “sum” on the questionnaire. She said “the sum of x multiplied by y is two” in order to interrupt herself and wondered why she written “sum” on both questionnaires. It was apparent that she now realized that what she had written was incorrect, but she does not recall the word “product” until I suggest it. A discussion on terminology for addition and multiplication followed. Vera stated that it was likely that the uncertainty had existed “since I learned to count” early in school. (The theory that knowledge gained during the early years of school is of significance during subsequent years has been presented by Helldén, 2004).

It was noticed that Vera applied a graphical interpretation to the statement in question 3, regarding the shape of a line, but not to the statements in questions 4 and 5:

I: Considering questions 3, 4 and 5, do you see any images before you, or... for the first question, I have got the impression that you saw a line.

V: Yes.
I: How was it for questions 4 and 5?
V: In question 4 I saw the area of a circle. When I see it in front of me, I see a round shape, is it a circle? Yes, that’s what it looks like. No, I see the area of a circle…and the radius and so on…although it doesn’t say “radius”. But I have explained that here [referring to \( a=\pi r^2 \) in the answer on the questionnaire]. What does a radius looked like? Yes, I can quite simply see the area in front of me…of a circle … I don’t see it as a function\(^{14}\).
I: Hmm, no.
V: Although I thought that it was. But I don’t see it like that, not at first glance.
I: No.
V: Concerning y … I don’t see anything in particular at all [referring to question 5].

It is clear that Vera could evoke mental images for \( y=x+5 \) and \( y=\pi x^2 \) in the form of a line and a circle respectively, but not for \( xy=2 \) (as was the case with Emma and Nils). I noted that Vera had previously mentioned that \( xy=2 \) could be written in another form \((x=2/y)\), to which she replied that one can either write \( xy=2 \) so that “x or y are free” and also stated that \( y=2/x \). In conjunction with that, she also considered whether \( y=2/x \) resulted in a “line”, saying that she was thinking about “the equation of the straight line”. Vera made no attempt to draw the graph, but instead followed up her assessment of straight lines by noting that the statement \( y=x+5 \) in question 3 results in a straight line since it has the form \( y=kx+m \), and stated that \( k \) is 1 and \( m \) is 5. She realized that the statement in question 5 cannot be written in the form \( y=kx+m \) and was therefore not a line.

Vera said “I don’t like that [statement in] number 5” and requested assistance with the interpretation of \( xy=2 \). I therefore asked her if she thought that the statement was more familiar in the form \( y=2/x \), but Vera did not think so, wondering if it ought to. She did not make any other associations. When I mentioned the concept of proportionality as an example of what the statement could represent, she became more enthusiastic, but was still uncertain as to how it should be linked to \( xy=2 \). It was revealed in the interview that Vera associated reverse proportionality with “\( x^2 \) and such things…” and appeared to relate to mathematical models in science.

I later raised the concept of inverse function, which Vera recognized but did not understand what it was. I asked whether \( x \) was a function of \( y \) in any of the questions 3 to 5. Vera stated that \( x \) is a function of \( y \) in \( y=x+5 \) and \( xy=2 \) after she had solved for the variable \( x \). Vera did not succeed in solving for \( x \) in \( y=\pi x^2 \) and was uncertain as to whether \( x \) was a function of \( y \). She subsequently realized that \( \pi x^2 \) is the area of a circle, where \( x \) is the radius; she did not think that it was a function, but could not explain why.

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\(^{14}\) Vera’s answers are not completely consistent with the answers she gave earlier in the interview in relation to question 2c, when she gave \( y=\pi x^2 \) as an example of a function (or a “dependency”).
It was observed that Vera mentioned a dependency when she said that the statements represented functions, but did not always refer to a dependency between the x and y variables; instead, she said that y is a function of πx² or x². It was also revealed in the interview that Vera associated x² with quadratic functions, which she most closely identified with y=x²; she could not decide whether y=πx² was a quadratic function, since it included π.

6 Discussion and conclusions

6.1 The function concept

Although the function concept is a key concept in the calculus course, none of the preservice teachers in the case study mentioned on their second questionnaire that the statements represented functions. On the other hand, formulations which highlighted the dependencies of variables were common (as in the case of Emma and Nils, who emphasized this by writing xy=2 in explicit form as y=2/x).

It was revealed that the preservice teachers did not describe a function in a way that is consistent with the definition of function. Nonetheless, Emma noticed an important component of the function concept, the univalence criterion, when she writes “for each value of x, there is just one value of y” on the first questionnaire; at the same time, she deviates from the definition by assuming that x and y are part of “an expression with two variables”. She commented on her answer to the questionnaire by stating “… the theoretical stuff which one has learnt, has been drilled into us…” and does not mention the univalence criterion on the second questionnaire, thereby showing signs of rote learning (Ausubel, 1968, 2000; Ausubel et al., 1978). In addition, Nils viewed the function concept as a dependence between variables. Like Nils, Vera based her conceptual interpretation of function on a dependency relation. But in contrast to Nils, she specified a dependency which did not exclusively consist of variables, but also included constants, in parallel with numerical calculations.

One explanation that the preservice teachers are unsuccessful in describing the function concept in a way that is consistent with the definition may be that they appeared to have little experience of working with problem formulations which require them to contemplate the definition of function (Even, 1993; Tall, 1996; Tall & Bakar, 1992; Vinner, 1992; Vinner & Dreyfus, 1989). The preservice teachers give the impression of only having been exposed to problems which encourage reflection on the function concept to a very small degree. This was revealed in the interview with Emma when she stated that she had not “come across” such situations. It was confirmed in the interview with Nils, who clearly could not recall the definition of function,
since he could not determine whether \( y \) was a function of \( x \) although he stated that a positive value of \( x \) results in two \( y \)-values for \( x=\pi y^2 \).

It can be stated that the function concept to a higher degree is tied to a structural conception (Sfard, 1991, 1992) in the form of ordered pairs in Emma’s case. The preservice teachers increasingly use an operational conception (Sfard, 1991, 1992) of function, identifying dependencies between the variables in the statements, as their level of performance in mathematics become lower; that in Veras’ case often highlights a process which is analogous to numerical calculations.

### 6.2 Function classes, properties and language

Williams (1998) states that mathematicians with PhDs often take note of various function classes and properties of functions when they draw concept maps\(^{15}\) for the concept of function. Slavit (1997) states that focus on the properties of functions may lead to a more developed structural conception (Sfard, 1991, 1992) in the form of objects. A comparison of the preservice teachers in the case study reveals that they did not stress that \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \) represent different function classes. Nevertheless, Emma stated that \( y=\pi x^2 \) was a quadratic function, both in her interview and on her concept map. In that respect, Nils referred to classes of curves (\( x^2 \)-curves and quadratic curves) and did not explicitly mention that he was referring to function graphs (although the concept map which Nils drew for \( y=\pi x^2 \) contains a horizontal sequence of nodes and links from the term “function” to “\( x^2 \)-curve” via “table” and “graph indicating that this was his intention). Vera seemed to associate quadratic function with \( y=x^2 \), but was uncertain as to whether \( y=\pi x^2 \) was also a quadratic function. Furthermore, they all stated that \( y=x+5 \) was a straight line (Emma stated that it was the equation of a straight line) and that it was a function, however none of the preservice teachers mentioned that it was a linear function. During their interviews, the preservice teachers also stated that \( xy=2 \) was a function (which they often preferred to write in the form \( y=2/x \)) but did not make any connections to a function class.

Emma and Nils leaned more towards a geometric (holistic) approach for \( y=x+5 \) and \( y=\pi x^2 \) in their interviews and on their concept maps, to a greater extent than in their answers to the questionnaires. This is also true of Vera in the case of \( y=x+5 \), while she largely viewed \( y=\pi x^2 \) as a formula for the calculation of an area, in contrast to \( xy=2 \), where none of the students gave a geometrical description without first plotting the values in order to sketch the graph. When the students discussed properties in conjunction with the function concept then they primarily did so from a geometric perspective (in Nils’

\(^{15}\) With the condition that the contents should be tied to a basic course in calculus.
case this is often combined with a numerical approach in which he refers to the values of the variables). A geometric approach is also reflected in the concept map that Emma drew for $y=\pi x^2$, in which she places “parabola” (referring to the function graph) in an underlying structure to “function”. To this, she tied “min. point” and “symmetrical”, among others. Emma further claimed that she now prefer to explain and illustrate problems with the aid of “images”.

When the preservice teachers are to describe the properties of functions, they often lack the mathematical language (Grevholm, 2000) that is related to the function concept. This was apparent when Nils said “rises completely” or when Emma said “roots of the equation where the function intersects the x axis” instead of using terms such as “increase” and “zero of the function” respectively. It was nonetheless shown that the preservice teachers used mathematical language, e.g. Emma mentioned the term “inverse function” in relation to the questions on whether $x$ was a function of $y$ and both Emma and Nils stated that $xy=2$ has asymptotes. The absence of a language reduces the possibilities for a concept to become well integrated in the cognitive structure (Ausubel, 2000). The preservice teachers’ language deficiencies with respect to different properties of functions may risk to prevent the construction of well-developed knowledge structures related to the function concept. This could create poor conditions for meaningful learning (Ausubel, 1968, 2000; Ausubel et al., 1978) and learning with understanding (Hiebert & Carpenter, 1992).

6.3 Knowledge structures and relations between concepts

The concept maps illustrate the preservice teachers’ views on how the different concepts which the statements are deemed to represent are related to each other (Hansson, 2004). It can be said that the maps drawn by the three preservice teachers are different in character. (Grevholm, 2000, show that concept maps preservice students draw also preserve some individual characteristics.) Emma\textsuperscript{16} and Nils drew maps which largely connected different sections of the map. They contained more\textsuperscript{17} cross-links (Novak, 1998; Novak & Gowin, 1984) in contrast to Vera’s two maps, neither of which contained

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\textsuperscript{16} The concept map which Emma drew for $y=\pi x^2$ differs from that which she drew for $y=x+5$ in that it had no cross-links. This is because (according to her) she first drew the concept map for $y=x+5$ and thought that it was time-consuming. The map of $y=\pi x^2$ subsequently became a first draft which she did not develop during the remaining time of that which was allotted to the preservice teachers to draw their maps. The concept map which Emma drew for $y=x+5$ contains both the function concept and the equation concept. She also mentioned these during the interview, saying that she had had “problems with them” when she drew the maps. This indicates that the concepts evoked conflicting concept images during construction which contributed to the time-consuming experience for Emma.

\textsuperscript{17} With the exception of the map which Emma drew for $y=\pi x^2$. 

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cross links. This could imply that the knowledge structures which represent the concepts are more compartmentalized for Vera than for Emma and Nils. Furthermore, the maps which Nils and Vera drew tended to contain more trivial elements (e.g., “draw”, “Greek letter”, “car romeo”) while the maps drawn by Emma were more subject-specific. This implies that Emma has knowledge structures which contain more meaningful relations to mathematical concepts.

The function concept is a more integrated part of the concept maps which Emma and Nils drew than in those drawn by Vera (with the exception of the map which Emma drew for \( y=\pi x^2 \)). In the map which Emma drew for \( y=x+5 \), the function concept is placed in close relation to the equation concept. It can be noted that both function concept and equation concept exists in all of the maps drawn by the preservice students for \( y=x+5 \) (all three of them also includes \( y=kx+m \), thus referring to the equation of a straight line). This is in contrast to the concept maps for \( y=\pi x^2 \); Vera is the only one who discusses the equation concept (by the link “remind me of” to the node “quadratic equation”). The preservice teachers appeared to see it as if the relation between the concepts “function” and “equation” changed in relation to the two statements \( y=x+5 \) and \( y=\pi x^2 \).

The form of a statement appears to determine which concepts the preservice teachers decide that it represents. During his interview, Nils says that he “automatically” sees a “\( y = x \)-expression” as a function (and both Emma and Nils write \( xy=2 \) in the form \( y=2/x \) in relation to the function concept). It was also clear that Nils did not view \( y=\pi x^2 \) as representing an equation, since he stated in his interview that “you can write it as an equation” and considered to gather the variables on one side of the equal sign. (He also said that it would then become “extremely tricky” and did not appear to realize that the solution set of the equation coincided with the graph of the function.) Furthermore, Emma did not state that \( y=\pi x^2 \) represented an equation (she did not mention the equation concept in either the questionnaires, the concept map or the interview), as opposed to \( y=x+5 \) and \( xy=2 \). The relations between concepts which were represented in the concept maps also appear to have been influenced; for example, function was strongly related to equation in the concept map which Emma drew for \( y=x+5 \), while the concepts had no relations in the map which she drew for \( y=\pi x^2 \), on which there was no equation concept.

It can be stated that the preservice teachers hardly make use of subject matter knowledge gained from tertiary level in relation to the statements, with concepts and relations such as from the recently concluded calculus course. Concepts mentioned to which they have been exposed to in the teacher training program but possibly not earlier, were diophantine equations (Nils), asymptotes (Nils, Emma) and inverse functions (Emma). Furthermore, the pre-
service teachers did not make any connections to future teaching scenarios or
the students’ learning in connection to the statements.

6.4 Dominant elements in concept images

The interviews reveal that Emma and Nils both seem to use \( y=x^2 \) as a prototype when they draw the graph of \( y=\pi x^2 \) (Akkoc & Tall, 2002; Hershkowitz, Schwarz, 1997; Tall & Bakar, 1992). The concept map which Nils draws for \( y=\pi x^2 \) confirms that he sees \( y=x^2 \) as a prototype in relation to \( y=\pi x^2 \) (while Emma states on her map that \( \pi \) influences the shape of the graph, indicating that she bases her assumptions on \( y=x^2 \)). It can be stated that \( y=x^2 \) appears to dominate the concept images presented by Emma and Nils, for a quadratic function and parabola, and a “\( x^2 \)-curve”, respectively. However, it does not immediately lead them to draw a correct function graph of \( y=\pi x^2 \), as both of the preservice teachers make the mistake of making a vertical translation of the graph during the interview. This differs somewhat from results presented by Hershkowitz and Schwarz (1997) which show that prototype elements may have a beneficial effect on problem solving. Furthermore the statement \( y=x^2 \) appears to have a strong influence on Vera’s concept image for quadratic functions. She identifies \( y=x^2 \) as a quadratic function and is uncertain whether \( y=\pi x^2 \) is a quadratic function, since the factor \( \pi \) is included.

In relation to \( y=x+5 \) and a straight line, the interviews and the concept maps reveal that \( y=kx+m \) constitutes a dominant feature in the preservice teachers’ related concept images, and they state that \( m \) and \( k \) are the coordinates’ \( y \)-value for the point of intersection with the \( y \)-axis and the slope of the line, respectively. Vera, for example, related to \( y=kx+m \) when she investigated whether the graph of \( y=-2/x \) is a straight line. It can be observed that in relation to \( y=kx+m \), the preservice teachers seem to apply a general model without a specific example.

Another situation arises during the interview, in which Emma spontaneously raises the relation between equation and function in connection with \( xy=2 \) (“equation” and “function” are thought to be represented by \( xy=2 \) and \( y=2/x \), respectively). It appears that Emma finds it difficult to distinguish between the two concepts and invents expressions, such as “the equation of the graph”. The concept map for \( y=x+5 \) (which has double links between the two concepts) and her reply to the question 2c of the second questionnaire (which contain words such as “solution pair”) also show that the cognitive structures which are related to the concepts seem to be in requirement of integrated reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978). During the interview, her understanding of the relations between “the equation of a straight line” and “function” appeared to be a cognitive obstacle (Goldin, 2002; Thompson, 1994) that made it difficult for her to distinguish the con-
cept of equation and the concept of function. Nevertheless, Emma came to a better understanding of the function concept in the interview when she considered $y^2=x$ and then tried to determine if $y$ is a function of $x$. It appears as if the situation stimulated a feedback of the concept image to the concept definition, that in Emma’s case lead to a better understanding, with signs of progressive differentiation and integrative reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978).

6.5 Closing comments

It is well known that mathematics students in the first year of university often do not have a conceptual understanding of functions that is consistent with the definition, as is the case with the preservice teachers in this study (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Tall & Bakar, 1992; Thompson, 1994; Vinner & Dreyfus, 1989; Williams, 1998, to name a few). However, Even (1993) argues that preservice teachers ought to have a deeper understanding of the function concept, while Vollrath (1994) states that understanding of mathematical concepts and their relations are important aspects of teachers’ knowledge skills. This is closely related to what Schulman (1986) refers to as pedagogical content knowledge. It was found that the preservice teachers in the case study have a limited understanding of the function concept and that they still do not possess the skills that an inservice teacher should possess. This is also expressed in the way they use mathematical terminology in relation to the concept of function; similar results related to preservice students’ language can be found in Grevholm (2004a, 2004b).

It seems that the preservice teachers in the study need to be exposed to problem formulations concerning the relations between mathematical concepts, including problems that invites to reflection upon the definition of function. The preservice teachers do not appear to be experienced in working with such problems. One way of developing their understanding of the function concept could then be to stimulate a feedback of evoked concept image to the concept definition (as seems to be the case when Emma comes to a better understanding during the interview). This type of reconnection is according to Vinner (1992) primarily possible with problems which are not of the standard variety. Moreover, if the preservice teachers’ concept image of function was characterized by those examples of function with which they come in contact, and to a lesser degree by the formal definition of the concept (a premise supported by, e.g., Vinner, 1983, 1992; Vinner & Dreyfus, 1989; Tall, 1996; Tall & Bakar, 1992), this should result in an attention of those examples of functions preservice students encounter during mathematics courses. Furthermore, highlighting different properties of functions and their relations to other concepts could be one way of stimulating a structural conception of function (Slavit, 1997; Sfard, 1991, 1992), and help the preservice teachers to develop
more composite knowledge structures in relation to the function concept, that the preservice teachers in the study seem to be in need of.

References


Appendix
Nils

**Matematik**
- Funktioner
  - Variabler
    - $x$, $y$
  - Räta linje
    - Kan skrivas på form $y = mx + b$
  - Allmän form
    - $y = mx + b$
  - Räta linjens ekv.
    - $y = x + 5$
  - Koordinatssystem
    - Skär $y$-axeln vid 5
  - Exempel

**Funktionslära**
- Variabler
  - $x$, $y$
  - Grekisk bokstav $\pi$
  - Närmevärde: $3.14$

**Yta**
- $y = x^2$
- $y = x^2$
- $y = x + 5$
  - Positivt

**Kapitel**
- Tenta
  - Klarade den
  - Hand
  - Träkigt
Vera

$y = x + 5$
Frågeformulär till studerande på lärarutbildningen med inriktning mot matematik åk 4-9

Namn: ..........................................................................................................


1) Sätt kryss i den eller de rutor du tycker passar bäst.

<table>
<thead>
<tr>
<th>Innehåller en variabel eller flera variabler</th>
<th>Är en ekvation</th>
<th>Är ett algebraiskt uttryck</th>
<th>Är en funktion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x + 3 = 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x + 5y = 8$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^2 + 3x - 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a - 3b + c$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = x^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(45 + 86)/12$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = e^x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = 1/x$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2) Beskriv med dina egna ord vad du menar med begreppen:

a) Variabel

b) Ekvation

c) Funktion

d) Algebraiskt uttryck
3) Vi skriver $y = x + 5$. Vad betyder det?

4) Vad kan du säga om $y = \pi x^2$? Svara så utförligt du kan.

5) Vad kan du säga om $xy = 2$? Svara så utförligt du kan.
6) Vi skriver \( y = f(x) \). Vad kan du säga om det?

7) Beskriv i vilken omfattning du menar att funktioner är av betydelse i matematik. Motivera din åsikt.

8) I vilken utsträckning anser du att funktioner är närvarande i grundskolans matematik. Motivera din åsikt.