

# The Concept of Function - Critical Aspects Induced by Teaching and Textbook

Constanta Olteanu  
Oskarshamn Local Authority,  
SE-572 28 Oskarshamn, Sweden

and

Mona Holmqvist  
Kristianstad University College,  
SE-291 88 Kristianstad, Sweden

## ABSTRACT

Although students' understanding of the concept of function has been studied by quite a number of researchers internationally, there has not been a lot of research on the subject in Sweden. This paper is an attempt to analyse what the students are offered to learn about the concept of function both in the classroom and in textbooks and what the students actually learn.

The presentation is based on data collected while the same object of learning is treated in two classes, and it includes two teachers and 45 students. Among other things, the data consists of video-recordings of lessons and tests. In the analysis, concepts relating to variation theory have been used as analytical tools. From this perspective a fundamental role of teaching is to bring critical aspects of subject matter into focus. The study focuses on what the individual is doing and expressing in relation to the object of learning.

The new way of understanding the relation between learning and teaching made it possible to find that some of the critical aspects in students' learning are induced by teaching and textbook exposés. One of these critical aspects is the argument of the function, namely whether it is presented implicitly ( $y$ ) or explicitly ( $f(x)$ ).

**Keywords:** argument, functions, teaching, learning, experience, theory of variation, dimensions of variation

## 1. INTRODUCTION

Mathematical teaching and learning have been the focus for researchers with different purposes and from different theoretical perspectives [see e.g. 2, 7, 17]. Sfard (1991) developed a theoretical perspective called process-object-perspective from which to analyse the

role of mathematical concepts in mathematical thinking. In short, the concept can be considered as both a process and an object. Based on this perspective, the algebraic representation of a function may be interpreted in two ways: operationally, as a concise description of some computation, or structurally. Breidenbach, Dubinsky, Hawaks and Nichols (1992) argues that understanding of the function concept can be achieved by developing "the ability to construct processes in their [the students'] minds and use them to think about functions" (p. 247) and Schwartz & Yerushalmy (1992) argue that "the symbolic representation ... is relatively more effective in making salient the nature of the function as a process ... the graphical representation ... is relatively more effective in making salient the nature of function as entity" (p. 263). Moschkovich, Schoenfeld and Arcavi, (1993) argue that a student can use the function concept if he or she "knows which representations and perspectives are likely to be useful in a particular problem context and is able to switch flexibly among representations and perspectives as seems appropriate" (p. 74). Despite extensive research on the basis of process-object-perspective, there are several researchers [see e.g., 1, 6] who argue that the process-object perspective can not explain how there may be students who do not see that, for example, the functions  $f(x) = 3(x^2 - 4)$  and  $g(x) = 3x^2 - 12$  are identical while the operational process differs. In addition, there are researchers [see e.g., 5], who note that it is also difficult to explain students' difficulties to recognize the different representations of the same concept from this perspective.

Other researchers focused on students' misconceptions that are: function is an algebraic equation; function should be given by one rule; graphs of functions should be regular and systemic;  $y = c$  ( $c$  is a constant) is not considered to be the representation of a function [see e.g., 4, 8, 15, 18, 19].

The function concept has previously been studied based on two dominant schools: rationalist (locating knowledge primarily in the brain or head, with rational

thought processes as the means of producing knowledge) and empiric (seeing objects in the world as the prime source of knowledge, which humans can never fully comprehend but can come to terms with through experience of the world). The starting point of this paper is not based on any of these traditions; it is instead based on the variation theory [9, 10, 11, 12], which defines learning as experiencing something in a qualitatively new and more powerful way. One of the most important concepts is the variation in the ways students experience aspects of the object of learning they become aware of in their environment which can be analysed and described in terms of qualitatively different categories of experiencing. Among these categories, teachers can identify the important aspects for current understanding, possibly not in the same way as the teacher's own understanding but adequately powerful for current concerns.

## 2. THEORETICAL ASSUMPTIONS

A mathematical function is a mapping from one defined space to another, such as  $f: R \rightarrow R$  in which  $f$  maps the real numbers to the real numbers. If the function for example is  $f(x) = x^2 - 4$ , this simply means that the mapping from  $x$  to  $f(x)$ <sup>1</sup> is the process that squares  $x$  and subtracts 4. If we list a set of inputs, we can define the corresponding set of outputs. We can visualize a function by giving the argument,  $x$ , some modification,  $f(x)$ , and coming out where the instructions for this process are given by the expression in  $f(x)$ . A function defines a relationship between one variable on the  $x$ -axis of a Cartesian coordinate system and an operation on that variable that can produce only one value on the  $y$ -axis. Often a function is explicitly defined as a mapping between elements of an ordered pair:  $(x, y)$ .

In order to use the concept of function in several ways (the most common types are algebraic expressions, tables, and graphs), the students must discern the function as a whole. To do this it is necessary that the students discern the principal parts of a function (argument, how the function operates on the argument and the results of this operating). Thereafter it is necessary to emphasize different parts, examine different aspects, examine the same aspects repeatedly but in different ways and combine the aspects differently. In this way it is possible for students to give each aspect a new meaning, and give a new interpretation to the whole picture. To discern the aspects, variations must be experienced, either at the same point in time, or by remembering earlier related experiences [9, 10, 11, 12]. To experience a function is to experience both its meaning, its structure and how these two mutually con-

stitute each other. So neither structure nor meaning can be said to precede or succeed the other. If these aspects are not focused on in a teaching situation or in the textbooks, they remain critical in the students' learning. "Critical" here refers to critical difference in the participating learners' ways of mastering the object of learning.

The concept of experience has guided the analysis of teaching and learning in this study. Learning, which in this perspective is identical with the experience of variation, has been studied from two perspectives, namely the enacted and lived object of learning. The enacted object of learning is what appears in the classroom, and the teacher and the students constitute it jointly. The enacted object of learning is what the researcher can observe in the classroom. The way in which students understand the object of learning is the lived object of learning, and can be described by their results on tests or the students' questions in the classroom. The study focuses on the same content, but the analysis points out that the object of learning is offered in different ways in the two classrooms with regard to what critical aspects are stressed. The analysis of the data describes how the students discern the parts and the whole and the relation between them.

## 3. METHOD

The study was performed in an upper secondary school in Sweden during the spring term 2003. During this period, the teachers' teaching of the same course in mathematics, namely Course B, and the students' learning were observed. In both classes, the same textbook was used. Students and teachers were selected from the Natural Science Program. A total of 45 students (25 males, 20 females) and two teachers (Anna and Maria) took part in the study.

In this study a combination of methods for collection of data was used. The data was collected in 10 steps. The students took a diagnostic test after they had studied the Mathematics A (Step 1). I followed the lessons of the classes in Course B, videotaped them and made notes after each lesson (Step 2). The students took two tests during the Course B with the teacher and a diagnostic test after they had studied Mathematics B (Step 3–5). On the basis of the test results and the students' willingness to participate, 8 students (four students in each class) were selected (in co-operation with the teachers) for an individual session, a post-test (contained tasks dealing with concepts that the students needed to develop further) and an interview (Step 6–9). The teachers that took part in the study analysed the solutions of four students, and concluded what each student could improve in her or his knowledge, analysed the reasons for the unsatisfactory understanding of the student, described a lesson plan for the individual session that was needed to help the student and chose the aids and

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<sup>1</sup> The notation  $f(x)$  for a function was introduced by Euler in the eighteenth century, and is the most common notation today.

tools needed to assist the student. The teachers and Olteanu looked at the video sequences together and discussed them (Step 10).

Since both teachers taught the same contents and used the same textbook, it was possible to identify and describe differences between the teachers' teaching in relationship to the contents they taught about. Before, after and during the observed education, all pupils wrote four tests. Since the students solved the same exercises in different tests, it was possible to identify and describe differences in their experienced contents.

#### 4. RESULTS

##### The lived object of learning

The analysis of the tests shows that the students in the two classes have difficulties learning the concept of function. In spite of this, the students in Anna's class performed better than the students in Maria's class (see Table 1). The reason for this difference has to do with the way in which the students discern the function argument and the function value when a function is represented by an algebraic expression and by a graph. This phenomenon was observed when the students answered the following exercises during the course:

1. Calculate  $f(4)$  if  $f(x) = 12 + x$  (2003-02-28)
2. Calculate  $f(-5)$  if  $f(x) = 32 - x^2$  (2003-02-28)
3. Read  $f(2)$  from the graph: (2003-04-25)

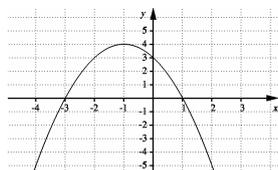


Figure 1. Function – graphic representation.

In these exercises, the arguments of the functions appear in explicit form and the questions focused on identify the value of the functions. The analysis of the students' answers is presented in the following table:

Table 1. Variations in how the students perceive the function argument and value.

The students' respons	Percentage of students in Maria's class		Percentage of students in Anna's class	
	algebraic (1: 2)	graphic (3:4)	algebraic (1: 2)	graphic (3:4)
The function arguments are identified with a given number or with a number on the x-axis	60 %	76 %	80 %	79 %
The function arguments are confused with the function value for this argument	20 %	24 %	5 %	14 %
The function arguments are maintained although there is a given value for x	15 %	0 %	10 %	0 %
Don't solve the tasks	5 %	0 %	5 %	7 %

The results show that a larger proportion of students in Anna's class discern the argument of the function and the relation between the argument and the value of the function both algebraically and graphically. From table 1 we can see that there are differences in the participating learners' ways of mastering the object of learning and that these differences have the argument of the function as basis.

A way of understanding why the students have difficult to discern the functions argument and the value of functions is to analyse the enacted object of learning.

##### The enacted object of learning

The analyses of video-recorded lessons show that the teaching of the concept of function involved activities that focused on the notation " $y, f(x)$  or  $y = f(x)$ " for a function. For example, in Maria's class and in the textbook, the graphical representation of functions which is noted as  $y = f(x)$ , is used with no explanation (see for example Figure 2). Maria only describes the procedure to identify the value of the function without focusing on the meaning of the used notations. In Anna's class these notations are presented separately, but she focuses on the fact that both notations have the same meaning (see Figure 3).

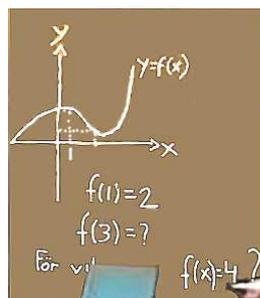


Figure 2. Notation of function (Maria)

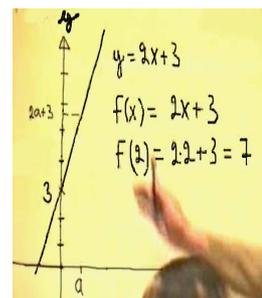


Figure 3. Notation of function (Anna)

Maria usually uses the notation  $y$  for a function (implicit argument) when she works with various functions that have an algebraic representation, whereas the notation  $f(x)$  (explicit argument) is used in Anna's class and in the textbook. Besides, Maria uses the concept of function ambiguously several times. An example of this is when she presents the points where a second-degree function is zero. For example, Maria uses the following system of equations to determine the zeros of a second-degree function:

$$\begin{cases} y = 0 \\ y = x^2 + 5x - 5 \end{cases} \quad (2003-04-04)$$

The way in which Maria uses this system of equations is presented in the following transcript:

[3] Maria: What I'm writing here is a system of equation. What do we want to find out in a system of equation? What does the equation

system, or its roots, tell us? What do we want to find out Sune?

- [4] Sune: Well, it's the expression we don't know.  $Y$  and how is it called...  $x$ .
- [5] Maria: Yes, well... what do we need, what is it we want the answer to? Rut?
- [6] Rut: Where the two lines cut ...
- [7] Maria: If the equation system consists of two lines, we want to know where they cut each other. Do we have two lines here?
- [8] The students are mumbling.
- [9] Maria: Yes, it is a line. What is it more?  $y$  equals 0, it's a line, but this time we don't have a line, it's a curve. The equation systems we've had earlier (Maria draws two lines cutting each other on the blackboard) but here we have an  $x^2$ -curve. What will it look like?
- [10] The students are silent.
- [11] Maria: What is that form called?
- [12] The students are still silent.
- [13] Maria: When we have an  $x^2$ -function. What is the form called, that curve, what do we call it?
- [14] Simon: A parabola ...
- [15] Maria: A parabola, yes (the teacher writes the word on the blackboard). (Lecture 26, 2003-04-04)

The conversation shows that Maria highlights that the given system of equations does not consist of two first-degree equations such as the students have previously experienced [6-7]. Instead, she focuses on the fact that the equations forming a system of equations also can contain algebraic expressions of the second-degree. A solution to a system of equations is an assignment of numbers to the variables so that all the equations are simultaneously satisfied. This means that the symbol  $y$  at this time is used to select an equation with two unknown variables. In addition, Maria focuses on the fact that the component of the system of equations consists of a function and a line [9-15]. This means that the symbol  $y$  at this time is used with multiple meanings, i.e. as an unknown quantity, to select an equation with two unknown variables and as a function. The impact of this is that the students don't have the possibility to discern if the presented exercise focuses on a function or an equation [13-14].

This ambiguous use of the meaning of the notation  $y$  or  $f(x)$  for a function is reflected when the students individually solve different exercises. An example of this is when Viveka (a student in Maria's class) tries to simplify the expression  $f(l+h) - f(l)$  with help of the function  $f(x) = kx + m$ .

[1] Viveka: I simplified it, so it's ... [...]

Viveka writes:  $f(l+h) - f(l) = f + fh - f$  on the paper.

[4] Maria: No, you must not do that because  $l+h$  is a single number. But you have  $f$  of  $l+h$  when  $f$  of  $x$  becomes  $kx + m$ . That means that  $x$  is equal to  $l+h$ .

[5] Viveka: Ok.

Anna systematically used the notation  $f(x)$  while working with various functions with an algebraic representation. So the students had the possibility to experience a variation in the way in which the principal parts of a function (argument, how the function operates on the argument and the results of this operating) relate to each other and to the function's concept as a whole. An example of this can be seen when Anna determine the zeros for the function  $f(x) = x^2 - 4x - 5$ :

[3] Anna: This is an equation, you see it as well as, instead of  $f$  of  $x$ , and  $y$  then becomes... (the teacher writes  $y$  above  $f(x)$  on the whiteboard). You see it like this. How many solutions does this equation have for starters? How many solutions does it have? Ulf?

[4] Ulf: Two.

[5] Anna: It has two... Does anyone want to debate this with Ulf now...  $y$  equals  $x$  square, minus four  $x$ , minus five... How many solutions does that equation have? Kurt?

[6] Kurt: Infinite ...

[7] Anna: Infinitely many, that is, all the pairs of  $x$  and  $y$  constituting [...] a graph. (Lecture 26, 2003-04-01)

The conversation shows that Anna focuses on the fact that the symbol  $y$  can be part of an equation, and can be used to select the points whose coordinates are  $x$  and  $y$  from the function graph [5-7]. In this way Anna opened up a dimension of variation in the argument of the function, namely that the argument of the function can be represented in explicit form and that the meaning of  $f(x)$  is a function, or in implicit form and then the meaning with the expression  $y = x^2 - 4x - 5$  can be a function or an equation.

## 5. CONCLUSIONS

One of the most important tasks of the teacher is to help his students to learn something. Trying to help the student effectively but unobtrusively and naturally, the teachers may be able to identify what must be focused on in a teaching situation in relation to the object of learning. In this article the object of learning is the concept of function. To use this concept in several ways, the students must be able to discern the function as a whole. To do this it is necessary for the students to discern the principal parts of a function (argument, how the function operates on the argument and the results of this operating). The results in this study show that the argument of the functions remains critical in students

learning despite the explanations in the classrooms and in the textbook. This phenomenon can be seen more in Maria's class than in Anna's class. One possible reason is the use of implicit arguments in Maria's explanations or using both implicit and explicit arguments without giving a clear explanation about the meaning of these arguments. This study points out that a good notation for function should not be ambiguous. It is inadmissible that the same notation denotes two different objects in the same inquiry (see Maria's presentation of the zeros of functions). We cannot use the same notation for different objects, but we can use different notations for the same object (see Anna's presentation of the zeroes of the function). The implication of an ambiguous notation does not give the students the possibility to experience the parts of the function simultaneously and by that the relation between these parts in a function are not discerned. If the students in a learning situation have the possibility to experience the meaning and structure of the function, they may be able to discern one representation from another or recognize that certain representations are functions and others not.

Teachers and authors of textbooks should not forget that the students have to know the motive and the purpose of various steps in solving a task. To make such steps comprehensible by suitable remarks or by carefully chosen questions and suggestions takes a lot of time and effort, but is worthwhile.

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