LEARNING ANALYSIS: STUDENTS' STARTING POINT

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Most concepts taught in a basic analysis course at university are somewhat familiar to the students. They use their representations from prior experiences to form new knowledge about the concepts. Fifteen students' were examined at the beginning of a basic university course in analysis. Their pre-knowledge was sometimes vague and even wrong, often due to intuitive perceptions.

A QUESTION OF PRE-KNOWLEDGE

A first step to learn something new, in mathematics as well as in other areas, is to find out what to start from. Students' pre-knowledge, which Ausubel (1968) so sturdily emphasised, determines in what pace to conduct the studies and at which level. A student who has a firm mathematical base from which he or she confidently manage to solve tasks and make sense of mathematical relationships can, in an insightful manner, learn new mathematical properties as they fit in with the already known properties. To learn means to understand which is different from just being able to perform a certain operation (Hiebert & Lefevre, 1986). A student who tries to learn from an insufficient base does not see the necessary patterns and properties to be able to structure the new, and old, knowledge in a manageable way.

Students' pre-knowledge was studied in an elementary analysis course comprising 15 students. The question investigated was: What relevant pre-knowledge do students have at the start of an analysis course? The aim was to unveil and categorise the students' different mathematical starting positions. The study is part of a larger study of changes in students' concept representations of analysis. One year after the beginning of the analysis course new data was collected to reveal the changes in the students' concept representations. Analysis of the new data is underway and will be presented later on.

INDIVIDUAL KNOWLEDGE REPRESENTATIONS

Each person has his or her own mental representation, or concept image (Tall & Vinner, 1981), of concepts and processes they have experienced. A concept image encompasses representations of concepts and processes learned or just briefly perceived. The representations are linked together. Derivatives and limits, for example, are linked by the limit definition. Tasks, algorithms and examples are also part of the concept image. Impressions from instructions, discussions, solving tasks and reading, which all lead to mathematical thinking, have impact on the development of the concept image.

When new parts are added, some rearrangements of the already existing concept image are sometimes needed. Reflection on what is already known is necessary

regardless of changes as new properties or objects are integrated. A concept image may comprise rote learned parts and parts that make no sense to the learner and even wrong parts. When something new is linked to an erroneous part the learner can become aware of the problem and corrects the error, but if he or she is unable to see the error or identify the link, a new part of the concept image can be created and exist in parallel with the erroneous part. The two parallels are evoked in different situations depending on the origins of their creations.

New concepts are sometimes introduced intuitively, perhaps with an image, to become more strict representations later on as the learner is able to link the intuitive representation to a stricter or a complete representation. Images of concepts can however work in the opposite of the intended way as Aspinwall, Shaw and Presmeg (1997) found in their case study on mental imagery. A person's mental images can confuse, rather than ease making sense of concepts and links between them, if their concept images do not cohere with formal concept definitions (Tall & Vinner, 1981).

Research expose students' struggle to link intuitive representations to formal representations (e.g. Dreyfus, 1991). Sirotic and Zazkis (2007) claimed that underdeveloped intuitions often are due to flaws in formal knowledge and an absence of algorithmic experience. Such a result was also drawn from a study on students learning limits of functions (Juter, 2006) where many students' intuitive perceptions were incoherent with the formal concept image leaving the students with two different representations, one for theory and one for problem solving. Links between intuitions, formal knowledge and algorithms are necessary for anyone to understand the topic at hand. Even if students do feel a need to link intuitive representations to new formal representations, they may recognise the efforts required to understand formally stated mathematics and settle with the intuitive representations which have been sufficient so far (Juter, 2006).

Hähkiöniemi (2006) has studied students learning the derivative and concluded that students could have a procedural understanding of the limit of the difference quotient, but still lack a conceptual understanding of the limit process. He stated that the students used a variety of different representations of limits in their learning of derivatives, but the students had difficulties to link them to formal mathematics. Viholainen (2006) studied other types of links and his results showed that the students in his study had difficulties to use concepts in the embodied world (Tall, 2004) in a constructive correct manner when they worked with continuity and differentiability.

To sum up, some students have an intuitive sometimes procedural conception of the concepts and need guidance to comprehend and formalise their knowledge. Concept images need to be coherent, general, clear, effective and unambiguous to allow productive mathematical progress.

A STUDY OF 15 STUDENTS' PRE-KNOWLEDGE

The students and the course

The aim of the study was to shed light on the different types of views of functions, limits, derivatives, integrals and continuity students brought to their first university analysis course. Student descriptions origin from their concept images at the particular moment of inquiry. Since concept images are in constant alteration, we obtain momentary glimpses of the students' concept images.

The 15 students in the study were attending an analysis course comprising limits, continuity, derivatives, and integrals as main topics. No selection of students was done since there were 16 students taking the course and one student was absent when the data was collected. Hence they performed at different levels mathematically. They were aged 19 years or older. Most of the students had the course as part of their teacher education programme, but it was also given outside the program. All students had met the concepts studied in this article at upper secondary school.

At the beginning of the course the students filled out a questionnaire where they were asked to describe the five, afore mentioned, concepts and also to write what the concepts are used for. The reason for this open approach was to prevent the students from becoming restrained with other formulations than their own. The questions used are presented in the results and analysis section.

Results and analysis

The results are exposed in tables 1-6. Tables 1-5 present the students explanations to functions, limits, derivatives, integrals and continuity, categorised according to type of explanation. Table 6 shows the students' descriptions of what the concepts are used for. The students are labelled with the letters A-O to retain anonymity. Within each category there can be both correct and incorrect descriptions. Typical examples are given below the tables.

Table 1 shows the students' responses to the first task: Describe the concept of function with your own words.

Table 1: Categories for students' descriptions of functions.

Category	Students in each category
1. Rule where one value becomes another value	CDEM
2. Relation between x and y	BFGHJN
3. Graph	AB
4. Description of an event	IKLO

The first category comprises students who regarded functions as something turning a value to a new value. Three of the four students stressed that the new value is different from the first (only student M did not). Student E wrote: "Insert a value get out something else". Students in the second category described a function as a relation between variables. A typical answer in this category was student G's: "A

relation where one x only can relate to one y ". Student F wrote: "Some things that are related to each other. If one has some value or values, the answer is uniform". In the third category, student B described a function as a "variable dependence which leads to some kind of graph". His answer is categorised in the second category as well. Student A stated that "a function is a formula like an equation with the difference that you can picture a function in a graph", showing a clear mix-up of functions and equations. The answers in the last category are exemplified by student L: "It is something describing a series of events of two different things, for example distance – time". Student I had a somewhat different description: "A function is a series of events shown by a formula or a coordinate system". The four types of descriptions all reflect different aspects of the formal concept definition of functions, with some misconceptions such as the perception that functions change all values.

The students' responses to the second task, which was: Describe the concept of limit with your own words, are shown in table 2.

Table 2: Categories for students' descriptions of limits.

Category	Students in each category
1. Closest value	CO
2. Derivative	D
3. Interval within which function values can be	ABFG
4. A number or infinity	HK
5. Maximum or minimum for a function	I
6. Stabilizing point	J
7. Mean value	N
8. No answer	ELM

Student answers in the first category reveal perceptions of limits as a value closest to the function at a certain point, for example student C: "A limit is the closest value a for a given number in a ". This student confused the limit with an interval which is a common way to see limits as the statements in category 3 reveal. The student in the second category remembered the notations used when dealing with derivatives, but was unable to make any sense of them. The third, and most frequently preferred, category comprises, as afore mentioned, statements viewing limits as intervals. Student F described limits: "Within what frame the answer can be". Students in this category described limits as intervals, but their descriptions imply that they may see the endpoints of the intervals as different limits, as opposed to the student example given in category 1. Students in the fourth category thought of limits as the outcome, the value: "It can be a determined value or infinity" (student H). Student I who stated that a "limit is either max or minimum values in the function" had an intuitively logical interpretation of the word 'limit' but not a mathematically useful interpretation. The student in category 6 described an oscillating function stabilizing at one value, the limit value. Student N in category 7 thought that a "limit is levelling out between values", i.e. some kind of mean value. This type of reasoning can be

similar to the student's reasoning in category 6 if the function oscillates over and under the limit symmetrically. Limits provided more types of perceptions than functions did. It is natural due to the complex nature of the concept. Categories 1, 4, 6 and possibly 7 partly cohere with the formal concept definition.

Table 3 shows the students' responses to the task: Describe the concept of derivative with your own words.

Table 3: Categories for students' descriptions of derivatives.

Category	Students in each category
1. Slope of a function	DGH
2. Rate of change	C
3. Values in an interval	EF
4. Means to simplify functions/equations	ABN
5. Means to find where a function is zero/max or min	IKO
6. Limit equals derivative	M
7. No answer	JL

A common way to describe derivatives was as the slope of a function. In category 1 student D described derivatives as "functions' slope at different places/times". In category 2, student C stated: "The derivative determines the rate of change at a certain time". Student E in category 3 connected derivatives to values in intervals: "Determine many values and then check off in a certain interval". Student F had a similar pattern of several parts, but confused it with a Riemann integral: "You divide the area within a graph in little parts to be able to determine the result at a certain value". The three students in category 4 saw derivatives as something to use to alter functions to more manageable functions, for example student N wrote: "You add several x etc in equations and similar (depending on how the equation looked like from the start)" and then on the question on what derivatives are used for: "You derive equations to more easily be able to determine the variables in the equations". There seems to be some kind of confusion with finding roots to polynomial equations with double roots or higher. The fifth category comprises student answers related to finding specific values, such as functions' extreme values. Student K wrote: "Derivative is when you derive a function and get, for example zero values". Student M in category 6 wrote: "Isn't derivative and limit, in principle, the same thing?". This is probably an effect of a weak concept image where definitions of derivative and limit are not successfully integrated. Only category 1, 2 and 5 are correct if compared to the formal concept definition.

The students' descriptions of integrals are presented in table 4 as they responded to the task: Describe the concept of integral with your own words.

Table 4: Categories for students' descriptions of integrals.

Category	Students in each category
1. Area under the graph	GHJKMO
2. Opposite to derivative	AKN
3. Connection to derivative	C
4. Study of limits	B
5. No answer	DEFIL

A majority of the students interpreted integrals as the area under the graph (category 1). Student K wrote: “An integral is a reversed derivative where you determine the area under a function at a certain distance”. This answer is categorised in the second category as well. The third category also links to derivative, but not as an opposite, as student C wrote: “Has a certain connection to derivatives”. Student B in category 4 simply wrote “Study of limits”. All responses cohere with the formal concept definition, but again, none of them are complete.

Table 5 displays responses to the question: What does it mean for a function to be continuous?

Table 5: Categories for students' descriptions of continuity.

Category	Students in each category
1. Even rate of change/development	ACE
2. Function connected and defined for all x	HKM
3. Function changes the same way all over	IL
4. The function has no limits	B
5. The function is not discrete	J
6. The function tends to infinity	F
7. No answer	DGNO

The first category comprises student answers depicting continuous functions as a smoothly developing graph, such as student E: “It develops evenly all the time”. The responses in this category are fairly intuitive. Students in the second category described continuous functions stricter: “It is defined for all x and connected” (student H). Student I described a continuous function in the third category: “It moves the same way all the time, for example the sine curve”. Student L wrote: “Every step at the x -axis gives the same difference at the y -axis”. Categories 4 to 6 each comprise one student answer, which is the category description. Categories 1, 2 and 5 can be part of the formal concept definition, particularly category 2.

We have, until now, looked at each concept separately, but to get a picture on what the students thought the concepts were used for, their answers on four tasks were combined in table 6. The tasks were: What are functions/limits/derivatives/integrals used for? Some repetition of the answers from the prior tasks occurred.

Table 6: Categories for students' descriptions of what the four concepts from tables 1 to 4 are used for.

Category	Function	Limit	Derivative	Integral
1. Examine graphs/functions	AEJ	AKO	ABDJ	N
2. Study developments of events	BCDFIKLMO	J	CGKMO	GO
3. Find function values (max/min or zero)	AC		HI	
4. Change functions to other functions			N	A
5. Differential equations		B		
6. Mean values			E	
7. Determine area or volume				BHK
8. Determine intervals of function values		CFGH		
9. Study relations	GHN			

The first four categories were used for more than one concept (the first two were used for all four concepts), whereas the rest only were used for one concept each. Students in the first category had a mathematical approach to their responses, without any context from real life. They wrote about exploring the functions. On functions, student A wrote: "To examine how different graphs behave". In the second category, on the other hand, the students linked to events and sometimes to examples. One such example, again on functions, was given by student M: "Solve problems of various kinds where values vary from dependent variables, for example, how far one can go in a certain time depends on the velocity". The third category links to exploring functions' values. Similar descriptions were given for functions and derivatives. Students A, B and N claimed that derivatives were something to use to simplify functions (table 3, category 4). This view was again apparent for students A and N in the fourth category in table 6. The answers in categories 5 and 6 were simply: "Differential equations" and "Check different mean values". The categories 7 to 9 are similar to those for respective concept in tables 1, 2 and 4. Particularly category 9 on functions resemble table 1, category 2 (students GHN are in both).

CONCLUSIONS

Intuitive influences on the students' concept images were apparent from, for example, the descriptions of limits and continuity. The closing in of the graph when dealing with limits suggests intervals, but the students in table 2, category 1 and 3, became confused by the intuitive representation in their concept formation, as described by Aspinwall et al. (1997). The students were influenced by the sense of evenness of continuous functions resulting in erroneous perceptions as presented in table 5. Categories 3, 4 and 5 show that the students saw a continuous function as a line and if x tends to infinity there will be no limits since the function tends to infinity. Methods used for problem solving also influenced concept images. Table 3, category 4, and table 6, category 4 show an example of such influence where the purpose of the method was unclear to the students as they used derivatives to simplify equations.

The students' showed links between concepts, many of them wrong though. There is hence a need to address every concept in connection to the others to prevent weak links between and within intuitive, procedural and formal representations as Hähkiöniemi (2006) and Viholainen (2006) found in their studies on derivatives. Knowing students' starting positions, teachers are better able to help them successfully link concepts together, which in turn is necessary for the prospective teachers among the students to be able to explain connections to their pupils.

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