Conceptualising Procedural Knowledge of Mathematics – Or the Other Way Around

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Abstract. We discuss some characteristics of mathematics in order to see why the procedural knowledge of mathematics dominates learning of mathematics, especially on tertiary level, and consider means to establish more relevant links between these two qualities of mathematical knowledge. Our focus is especially on the role of the language of mathematics. Some findings and experiences from an ongoing study and development project realized at the Department of Teacher Education at Savonlinna, University of Joensuu, are also reported.

Keywords: conceptual, language of mathematics, procedural.

1. Introduction

Let us consider the following two problems that were presented for a group of prospective mathematics teachers on an introductory course of real analysis. In Finland, this kind of a course usually belongs to the second year curriculum in mathematics teacher training.

Problem A. Show that in an ordered field it holds that $0 < x < y \Rightarrow 0 < x^2 < y^2$.

Problem B. Study the monotonicity of the function $f : (0,\infty) \rightarrow (0,\infty)$, $f(x) = x^2$.

Now, one might expect that, in general, students had succeeded much better in Problem B than in A because, out of these two closely related problems, B is more illustrative and put forward in a more familiar language than A. However, the evidence gathered from the above mentioned course (organised three times at our department during the last six years, approximately 15 students attended the course each time) proves, indeed, the opposite. Tragicomically, the correct answer to Problem A is also the best possible and the most complete answer to Problem B – in the language of mathematics.

Of course, this result can be marginalised by explaining it with the local instructor's great enthusiasm for abstract algebra and its influence on his teaching. Nevertheless, it can also be taken as another evidence for the fact that has been widely reported around the world: the procedural part dominates the conceptual part in the students knowledge of mathematics and its learning. By the procedural and conceptual knowledge of mathematics we mean in this note the same as the authors in Haapasalo & Kadijevich (2000), see also Hiebert & Lefevre (1986). More precisely,

> *Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only the knowledge of the object being utilized, but also the knowledge of formal and syntax for the representational system(s) expressing them.
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> *Conceptual knowledge* denotes knowledge of and a skilful drive along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

In general, the partition of mathematical knowledge into these components is a nontrivial problem and many descriptions of these concept have been given, cf. Haapasalo & Kadijevich (2000). Some researchers (e.g. Star 2002, 2005) has critisised the whole division, for instance, by pointing out that a certain kind of conceptual knowledge, not included in the original definition given by Hievert and Lefevre (1986), is inseparable from the capacity to use procedural knowledge in efficient ways.

However, in contrast to the above critics, we find Haapasalo's and Kadijevich's definitions of these concepts a dynamic framework to study learning processes related to higher mathematics since, for example, algorithms can also be seen as nodes and links in a graph representing the individual's conceptual knowledge of mathematics. Obviously, there are also other approaches to study the development of knowledge of mathematics. We acknowledge, for instance, the process-object duality of conceptions (Sfard 1991), the APOS-theory (e.g. Asiala et al. 1997), and the distinction of *concept image* and *concept definition* (Tall & Vinner 1981).

Previous studies have already revealed, among other things, that students' misconceptions, attitudes and experiences related to mathematics guide severely their concept learning (e.g. Hannula 1998) and that, in problematic situations, the strategies students most probably resort to usually neglect the conceptual knowledge related to the given problem (e.g. Lithner 2003). Further, students prefer to reflect on using the concept images rather than the concept definition (e.g. Vinner 1991).

In this note, we shall try find a new point of view to the topic by discussing first how the dominance of procedural knowledge over conceptual knowledge of mathematics, e.g. students' better success in Problem A than in B, can be partly due to mathematics itself. Namely, it is our hypothesis that, especially on the tertiary level, the dominance of procedural knowledge is characteristic of mathematics itself or, more precisely, the language of mathematics. From this viewpoint, we shall consider possibilities to establish more links between relevant procedural and conceptual knowledge. This will be done in the framework of the APOS-theory (Asiala et al. 1997). In addition to that, we shall also report on some experiences from an ongoing development project that have been realized at the Department of Teacher Education in Savonlinna.

2. Conceptual and procedural knowledge on the language of mathematics

In modern mathematics, the very central concept of (real) number is defined by a set of axioms. These axioms are, by the above definition, highly procedural by nature. Consider, for example, the following excerpt of an axiomatic statement that verifies that, in Abelian group, every inverse is unique.

$$y = y*e = y*(x*z) = (y*x)*z = e*z = z.$$

Indeed, most mathematicians can easily detect that the above line establishes the essential part of the proof for uniqueness of an inverse *without having any precise knowledge of the meaning of the symbols appearing in this expression.* This is characteristic of a large part of modern mathematics: during the last one hundred years, there has been a strong tendency to represent mathematical theories in a form where all knowledge ultimately follows from fundamental, basically set theoretical, definitions and relations between them (cf. Spengler 1923, Section 'Vom Sinn der Zahlen'). Thus this knowledge is highly procedural by nature: it must be deriveable from the fundamental definitions and axioms by a finite sequence of logical steps. In addition to that, these axioms and definitions have induced a grammar-like tradition to guide argumentation in mathematical communication. This is a strength of the language of mathematics: one who is familiar with the tradition can even gain semantic information relayed by a sequence of mathematical symbols already by only looking at its syntax.

Hence, and as the above exemplar proof shows, a sufficient knowledge for successful communication in the language of mathematics, indeed, can be purely procedural by nature. Therefore, it is not any longer surprising that the students' success in Problem A does not guarantee the success in Problem B. The complete solution of A in the language of mathematics does not require the conceptual knowledge that is necessary for B, although the solution of A can be conceptualised so that it turns into a solution of B, too. Another observation gained from the same study also illustrates quite well the procedural efficiency of the language of mathematics: there were students who were able to manipulate logical sentences with several quantifiers correctly but not able to do the same for analogous statements in their own mother tongue.

The students' results with Problems A and B also indicate that, regardless the explicitly stated definitions and introductory sections, an essential part of the conceptual knowledge related to the language of mathematics often seems to be underlying between the lines in mathematical texts, or more precisely, is noticed only by those who possess the sociomathematical norm (Yackel & Cobb 1996). As it appears, it is a nontrivial task to unify the available and sufficient procedural (e.g. the answer to A) and conceptual knowledge (e.g. the definitions of 'function' and ‘monotonicity’ which, of course, were underlined with several examples during the course) in a successful way.

Again, this phenomenon can be tracked down to the nature of the language of mathematics. Seen from the utmost, Bourbakist, viewpoint, the conceptual knowledge related to the axiomatic representations of mathematics can lie mainly in a dimension of applications. Hence, when practising mathematics using its own language, the conceptual understanding can hardly precede the procedural knowledge. (For a detailed discussion on the order of the development of conceptual and procedural knowledge of mathematics on primary and secondary levels, see Haapasalo & Kadijevich (2000) and the references given there.) For example, the understanding of the above mentioned concepts that are necessary in Problem B requires a wide range of procedural knowledge related to, for instance, the manipulation of sets and orders.

As a matter of fact, the above assertion on the order of the development of conceptual and procedural knowledge is also backed up by the APOS-theory. That a concept encapsulates into an object and eventually a scheme, it requires that it has already been understood as a process (or a set of processes) on which one can perform transformations (Asiala et al. 1997). More precisely, the learning of a concept (or using Sfard’s terminology, a concept definition) and its usage in the language of mathematics requires sufficient amount of relevant procedural knowledge on which the encapsulation can happen.

By the above observations, it now seems plausible that some of the dominance of the procedural knowledge in teaching and doing of mathematics can be eventually passed down from the language of mathematics itself. Hence we are led to the following pedagogical conclusion: the reinforcement of the conceptual part of students’ knowledge of mathematics can happen by building more links within the procedural knowledge that is relevant to the concepts to be studied in such a way that takes into account the tradition and the sociomathematical norm that underlie the language of mathematics. Doing so, the students’ procedural knowledge will gain more meaning by gathering more applicability. In other words, it conceptualises.

In the next section, we shall discuss an effort to build more such links. Due to the limited space, unfortunately, we can only scratch the surface here.

3. Concepts as procedures

The concept of limit is known to be troublesome for mathematics students all around the world. In the language of mathematics, for example, the definition of the limit of sequence, however, can be expressed in a compact and seemingly simple way. Namely, the sequence (a_n) converges to the limit a if (and only if)

$$\forall \varepsilon > 0 : \exists n_\varepsilon \in Z_+ : \forall n \in Z_+ : n \geq n_\varepsilon \Rightarrow |a_n - a| < \varepsilon.$$

Of course, the content of the definition can be expressed using only natural language. Practically in every university textbook, nontheless, the main point of the definition is that a sequence converges to the limit if a certain statement is true.

The challenge the students have to face with the definition is that, even though there are several dynamic processes related to limit, the above logical expression (or a corresponding statement in natural language) on its own is totally static: for every (a_n) and a, it is universally true or false and nothing else. Moreover, the procedural knowledge related to manipulation of logical or set theoretical expressions is different from the one related to manipulation of sequences. Therefore, one can say that comprehending the definition does not yet guarantee the ability to determine whether a given sequence converges or not, and if it does, then what the limit is. Traditionally, this kind of procedural knowledge is taught by introducing students with a set of convergence theorems and a variety of examples. Unfortunatelly, approaching this way does not necessarily result to links between the definition and the relevant procedural knowledge that were strong enough.

In the development project in Savonlinna, one attempt to link the conceptual knowledge related to definitions of mathematical concepts to the procedural knowledge that is relevant for studying concrete examples was to ask students to translate the definitions to be studied into a form of algorithm or list of subtasks whose solving determines whether a studied object has the property of the definition or not. The motivation for the translation is to emphasise the process nature of concepts: to unwrap the structurally described definition to the language that is more compatible with the operational nature of concept or, in other words, to reverse the outcome of reification back to its raw material (cf. Sfard 1991; Sfard & Linchevski 1994). This approach is backed up by Sfard's ideas on concept learning: the operational conception is, for most people, the first step in the acquisition of new mathematical notions (Sfard 1991). In other words, the ultimate objective is to learn to comprehend the language of mathematics.

For instance, in the case of the limit of sequence, the algorithm might be such that as input one can give any sequence (a_n) and a real number a and as output one gets 'yes' or 'no' depending on whether a is the limit of (a_n) or not. The following example, which is a distillation of both students' and the author's ideas of the concept, illustrates the spirit of the approach.

Example. The real number a is the limit of the sequence (a_n) if the output of the following procedure LIMIT is 'yes'.

Begin LIMIT.

A1. Choose (small) $\varepsilon > 0$.

A2. Find out whether there is a number n_ε so that the distance between a_n and a is less than ε for all a_n for which $n > n_\varepsilon$.

A3. If the answer in A2 is 'no', then the output of LIMIT is 'no'.

A4. If the answer in A2 is 'yes', find out whether there is a function $N_\varepsilon : (0, \propto) \rightarrow N, N_\varepsilon(\varepsilon) = n_\varepsilon$ so that the distance between a_n and a is less than ε for all a_n for which $n > n_\varepsilon$.

A5. If the answer in A4 is 'no', then the output of LIMIT is 'no'.

A6. If the answer in A4 is 'yes', then the output of LIMIT is 'yes'.

End LIMIT.

In the algorithm, the main motivation for the presence of steps A1-A3 is, of course, the possibility of divergence of (a_n). However, they also serve another purposes: first, they help students to focus on the core problem (A4) and, second, since simpler to treat than A4, to strengthen their self-confidence by giving them a feeling of succeeding and getting into the swing of thing. Moreover, for the sake of simplicity, we have avoided the IF-THEN-ELSE-structure and used instead of that the IF-THEN-sturucture twice, cf. A5 and A6.

In general, the language used in the description of the subtasks plays an essential role. Here we have tried both to use genuine language of mathematics and, on the other hand, to rely only on those concepts that are most familiar to students: 'function' and 'distance' since the concept of function was studied in detail on a preliminary course and measuring distances is a well-known and illustrative process, cf. Figure 1 below.

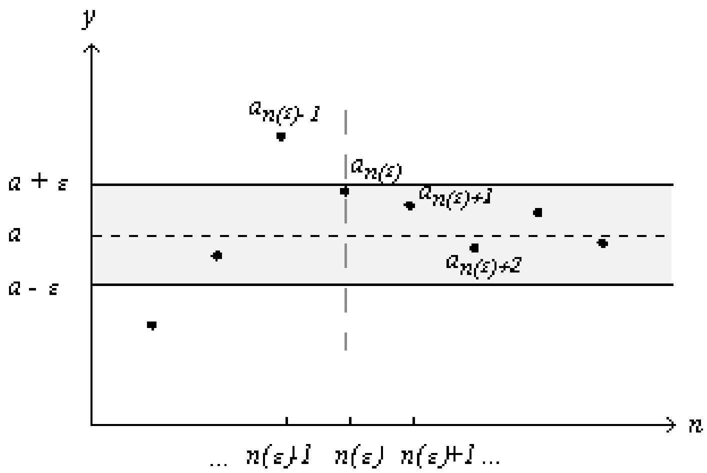

Figure 1. The illustration of the limit of the sequence.

Observe also that the subtasks A2 and A4 have been set up so that verbally they are more compatible with the classical convergence theorems than the original definition. For example, answering negatively to A2 provides a classical criteria for the divergence of sequence. Further, using the pinching theorem for sequences is just applying A4 for a majorant and a minorant sequences.

The experiences gathered from the development project in Savonlinna so far indicates that students do benefit from this kind of algorithmic approach. Before the project started, approximately one third of the students learned to treat simple sequences by the definition and one third of them were practically fully helpless. During the project, the propotion of those who quite plausibly seem to master the definition has raised to more than a half and, simultaneously, the number of totally helpless students has dropped. However, the thorough analysis of the results is not yet fully completed.

4. References

Asiala, M., Cottrill, J., Dubinsky, E. and Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of mathematical behavior*, 16 (4), 399-431.

Haapasalo, L. & Kadijevich, Dj. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*, 21 (2), 139-157.

Hannula (Ed.), M. (1998). Current State of Research on Mathematical Beliefs V. Proceedings of MAVI-5 Workshop August 22-25, 1997. Reseach Report 184. Helsinki: Department of Teacher Education, University of Helsinki.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.) *Conceptual and procedural knowledge: the case of mathematics*, (pp. 1-27). Hillsdale (NJ): Lawrence Erbaum.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics,* 52, 29-55.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on prosesses and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, 1-36.

Sward, A. & Linchevski, L. (1994). The gains and the pitfalls of reification – the case of algebra. *Educational Studies in Mathematics,* 26, 191-228.

Spengler, O. (1923). *Der Untergang des Abendlandes: Umrisse einer Morphologie der Weltgeschichte*. München: Beck. (47th Edit. 1990).

Star, J.R. (2000). Re-"conceptualizing" procedural knowledge in mathematics. In M. Fernandez (Ed.), *Proceedings of the twenty-second annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 219-223). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Star, J.R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36* (5), 404-411.

Tall, D. & Vinner, S. (1981). Concept image and concept definiton with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12, 151-169.

Vinner, S. (1991). The role of Definitions in teaching and Learning. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht: Kluwer.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27 (4), 458-477.